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Comprehensive Room Transfer Functions for Efficient Calculation of the Transient Heat Transfer Processes in Buildings

This paper describes a method in which the transfer functions describing heat flows in building elements can be combined into a single transfer function for an enclosure, referred to as a comprehensive room transfer function (CRTF). The method accurately models long-wave radiation exchange and convection in an enclosure through an approximate network, referred to as the "star" network. Resistances in the star network are determined from a network that uses view factors to model long-wave radiation exchange. The Padé approximation and bilinear transformation are used to reduce the number of coefficients in a CRTF.

Introduction

Year-long simulations of the heating and cooling loads of buildings are important for sizing heating, ventilating, and air conditioning equipment, determining the effect of a design change or retrofit on energy usage, and developing optimal control strategies. Accurate calculation of building loads involves the long-time solution of transient conduction, convection, and radiation heat transfer processes. Calculation of these loads requires significant computational effort.

Transfer function methods are more efficient for solving long-time transient heat transfer problems than Euler, Crank-Nicolson, or other classical techniques. Transfer functions relate the output of a linear, time-invariant system to a time series of current and past inputs, and past outputs. Inputs are modeled by a continuous, piecewise linear curve or equivalently, a series of triangular pulses.

The definition of transfer function used in the field of heat transfer in buildings is different from that used in the field of automatic controls. In automatic controls, a transfer function is the Laplace or z transform of the output divided by the Laplace or z transform of the input. In heat transfer, a transfer function is a difference equation that relates the outputs of a linear, time-invariant system to a time series of current and past inputs, and a time series of past outputs. In this paper, the latter definition will be used. Also, this paper uses Laplace transfer function as the definition for the Laplace transform of the output divided by the Laplace transform of the input and z-transfer function as the definition for the z transform of the output divided by the z transform of the input.

There are a number of methods available for calculating transfer functions. Stephenson and Mitalas (1971) present a method for determining transfer functions for one-dimensional heat transfer through multilayered slabs by solving the conduction equation with Laplace and z-transform theory. Mitalas and Arsenault (1971) wrote a program for computing transfer function coefficients based upon the method of Stephenson and Mitalas. Transfer function coefficients computed from Mitalas and Arsenault's program for 40 roof, 103 wall, and 47 interior partition constructions are listed in the ASHRAE Handbook of Fundamentals (1977, 1981, 1985). Ceylan and Meyers (1980) and Seem et al. (1987) present methods for calculating transfer function coefficients for multidimensional heat transfer. A discussion of two methods that use transfer functions to calculate building loads follows.

ASHRAE (1977, 1981, 1985) discusses the energy balance method (ASHRAE refers to this as the heat balance method) for calculating sensible heating or cooling loads for buildings. An energy balance equation is written for every surface in a room and the room air. For a room with N surfaces, these energy balance equations can be formulated in the matrix equation.

$$\mathbf{AT}=\mathbf{B} \tag{1}$$

where $\mathbf{A}$ is an $(N+1)$ by $(N+1)$ matrix, which contains transfer function coefficients, convection coefficients, and linearized long-wave radiation resistances; $\mathbf{T}$ is a vector of temperatures with all rows equal to an interior surface temperature, except the last row, which is the room air temperature; and $\mathbf{B}$ is a vector of current inputs, past inputs, past interior surface temperatures, and transfer function coefficients. After solving the matrix equation for the $\mathbf{T}$ vector, the load due to convective heat transfer between surfaces and the room air can be calculated. The computational effort is reduced when the $\mathbf{A}$ matrix is time invariant, i.e., convection coefficients and linearized long-wave resistances are constant. Mitalas (1965) shows that the cooling loads for a room are quite insensitive to changes in the interior convection coefficient. Walton (1980) shows that long-wave radiation exchange between surfaces in a room can be linearized without introducing significant errors. Assuming long-wave radiation resistances are constant does not introduce significant errors because the average temperature of surfaces in a room is fairly constant. Estimating exterior convection coefficients that vary with time is difficult due to the large number of factors (e.g., building size, building shape, building surrounding, wind direction, local wind velocity) that affect convective heat transfer from exterior surfaces.

Madsen (1982) develops a comprehensive room transfer function (CRTF) by using linear regression on results from an energy balance simulation. A CRTF is a single transfer function equation for computing loads or floating indoor air temperatures in a room or zone. CRTF simulations require less computational effort than energy balance simulations because only outputs of interest are computed.

This paper describes a method in which the transfer functions describing heat flows in building elements can be combined into a CRTF for an enclosure. The first section of this paper presents the derivation of the equations for combining

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transfer functions with parallel heat transfer paths. The second section of this paper is devoted to a method for accurately modeling the convection and radiation heat transfer processes in an enclosure by a star network. A star network allows individual transfer functions for building elements to be easily combined for rooms. The third section of this paper presents equations for combining transfer functions for a room model based on a star network. A method that can be used to reduce the number of coefficients in a transfer function equation is presented in the fourth section. The last section of this paper compares the computational effort of energy balance simulations with CRTF simulations that use reduced coefficients.

Parallel Path Combination

Equations for combining transfer functions for walls with parallel heat transfer paths, as shown in Fig. 1, are derived in this section. The transfer function equation for the heat flux for wall a is

$$q''_{t,a} = \sum_{n=0} (a_{n,a} T_{t-n\delta,o} + b_{n,a} T_{t-n\delta,i}) - \sum_{n=1} (c_{n,a} q''_{t-n\delta,a}) \quad (2)$$

The upper limits on the summations in equation (2) are dependent upon the method used to obtain the transfer function coefficients.

Equation (2) can be expressed in terms of the backshift operator B as

$$\left(\sum_{n=0} c_{n,a} B^n\right) q''_{t,a} = \left(\sum_{n=0} a_{n,a} B^n\right) T_{t,o} + \left(\sum_{n=0} b_{n,a} B^n\right) T_{t,i} \quad (3)$$

where $c_{o,a} = 1.0$. The backshift operator (Box and Jenkins, 1976) is defined as

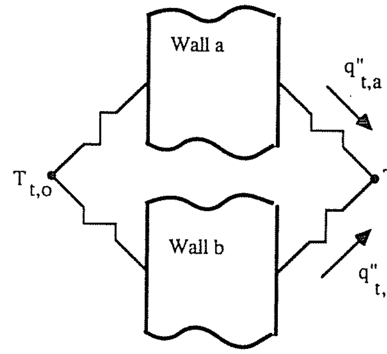

Fig. 1 Walls with parallel heat transfer paths

$$Bz_t = z_{t-\delta} \quad (4)$$

Dividing by the summation on the left side of equation (3) results in

$$q''_{t,a} = \frac{\left(\sum_{n=0} a_{n,a} B^n\right) T_{t,o} + \left(\sum_{n=0} b_{n,a} B^n\right) T_{t,i}}{\left(\sum_{n=0} c_{n,a} B^n\right)} \quad (5)$$

A similar equation for wall b is

$$q''_{t,b} = \frac{\left(\sum_{n=0} a_{n,b} B^n\right) T_{t,o} + \left(\sum_{n=0} b_{n,b} B^n\right) T_{t,i}}{\left(\sum_{n=0} c_{n,b} B^n\right)} \quad . (6)$$

Nomenclature

a = outdoor temperature transfer function coefficient for a wall; input transfer function coefficient

$\ddot{a}$ = input coefficient for Laplace transfer function, input coefficient for w transfer function

A = area

A_w = window area

$\mathbf{A}$ = $(N+1)$ by $(N+1)$ matrix, equation (1)

b = indoor temperature transfer function coefficient for a wall; output coefficient for transfer function

$\bar{b}$ = input coefficient for Laplace transfer function, input coefficient for w transfer function

B = backshift operator, equation (4)

$\mathbf{B}$ = $(N+1)$ vector of inputs, equation (1)

c = transfer function coefficient for past heat fluxes for a wall, input coefficient for reduced transfer function

$\check{c}$ = coefficient for power series expansion of Laplace transfer function

d = outdoor temperature transfer function coefficient, output coefficient for reduced transfer function

$\bar{d}$ = input coefficient for reduced Laplace transfer function coefficient, input coefficient for reduced w transfer function

e = star or indoor air temperature transfer function coefficient, output coefficient for reduced transfer function

$\ddot{e}$ = output coefficient for reduced Laplace transfer function, output coefficient for reduced w transfer function

f = solar radiation transfer function coefficient

f_a, f_b = area fraction for walls a and b, respectively, equation (7)

g = radiation transfer function coefficient

h = transfer function coefficient for past outputs

G_{i-j} = absorption factor between surfaces i and j, equation (15)

$G(s)$ = Laplace transfer function in Appendix A

$G_r(s)$ = reduced Laplace transfer function in Appendix A

I = incident solar radiation

N = number of surfaces in enclosure or room

R = resistance between star node and room air, equation (28)

R_{i-j} = resistance between surfaces i and j in the view factor network, when other nodes are floating, equation (20)

R_{i-r} = resistance between surface i and the room air in the view factor network when other nodes are floating, equation (22)

R_k = resistance between interior surface of wall k and star node, equation (23)

The heat flux for walls a and b combined, $q''_{t,x}$, is

$$q''_{t,x}=\frac{A_a}{A_a+A_b}\,q''_{t,a}+\frac{A_b}{A_a+A_b}\,q''_{t,b}$$
$$=f_a q''_{t,a}+f_b q''_{t,b} \tag{7}$$

Substituting equations (5) and (6) into equation (7) gives

$$q''_{t,x}=f_a\left[\frac{\left(\sum_{n=0} a_{n,a}B^n\right)T_{t,o}+\left(\sum_{n=0} b_{n,a}B^n\right)T_{t,i}}{\left(\sum_{n=0} c_{n,a}B^n\right)}\right]$$
$$+f_b\left[\frac{\left(\sum_{n=0} a_{n,b}B^n\right)T_{t,o}+\left(\sum_{n=0} b_{n,b}B^n\right)T_{t,i}}{\left(\sum_{n=0} c_{n,b}B^n\right)}\right] \tag{8}$$

Multiplying by the denominators of the terms on the right-hand side of equation (8) results in

$$\left(\sum_{n=0} c_{n,a}B^n\right)\left(\sum_{n=0} c_{n,b}B^n\right)q''_{t,x}$$
$$=\left[f_a\left(\sum_{n=0} a_{n,a}B^n\right)\left(\sum_{n=0} c_{n,b}B^n\right)\right]T_{t,o}$$
$$+\left[f_b\left(\sum_{n=0} a_{n,b}B^n\right)\left(\sum_{n=0} c_{n,a}B^n\right)\right]T_{t,o}$$
$$+\left[f_a\left(\sum_{n=0} b_{n,a}B^n\right)\left(\sum_{n=0} c_{n,b}B^n\right)\right]T_{t,i}$$
$$+\left[f_b\left(\sum_{n=0} b_{n,b}B^n\right)\left(\sum_{n=0} c_{n,a}B^n\right)\right]T_{t,i} \tag{9}$$

Carrying out the algebra and combining common powers of the backshift operator yields

$$\left(\sum_{n=0} c_{n,x}B^n\right)q''_{t,x}=\left(\sum_{n=0} a_{n,x}B^n\right)T_{t,0}+\left(\sum_{n=0} b_{n,x}B^n\right)T_{t,i} \tag{10}$$

where

$$a_{n,x}=\sum_{i=0}^{n}\left(f_a a_{i,a}c_{n-i,b}+f_b a_{i,b}c_{n-i,a}\right) \tag{11}$$

$$b_{n,x}=\sum_{i=0}^{n}\left(f_a b_{i,a}c_{n-i,b}+f_b b_{i,b}c_{n-i,a}\right) \tag{12}$$

$$c_{n,x}=\sum_{i=0}^{n}\left(c_{i,a}c_{n-i,b}\right) \tag{13}$$

Using the definition of the backshift operator, equation (10) can be rewritten in a form that looks exactly like equation (2)

$$q''_{t,x}=\sum_{n=0}\left(a_{n,x}T_{t-n\delta,o}+b_{n,x}T_{t-n\delta,i}\right)-\sum_{n=1} c_{n,x}q''_{t-n\delta,x} \tag{14}$$

where equations (11), (12), and (13) define the transfer function coefficients. The number of previous time steps in the combined transfer function equation is equal to or greater than the number of previous time steps for each individual transfer function.

Nomenclature (cont.)

$R_{i-j,\mathrm{rad}}$ = resistance to radiation exchange between surfaces i and j, equation (15)
$R_{i,\mathrm{conv}}$ = resistance to convective heat transfer between surface i and room air
$R_{k,\mathrm{out}}$ = resistance to convective heat transfer between exterior surface of wall k and outdoor air
s = complex Laplace transform variable
T = temperature
$\bar{T}$ = average temperature of surfaces in a room, equation (15)
$\mathbf{T}$ = $(N+1)$ vector of temperatures, equation (1)
u = input
w = complex variable that results from bilinear transformations
$x_{(i,j)}$ = element in row i and column j of the $\mathbf{X}$ matrix, equation (18)
$x_{(i,j),\mathrm{inv}}$ = element in row i and column j of the inverse of the $\mathbf{X}$ matrix
$\mathbf{X}$ = $(N+1)$ by $(N+1)$ matrix, equation (18)
$y_{(i)}$ = element in row i of $\mathbf{Y}$ vector, equation (18)
y = output
$\mathbf{Y}$ = $(N+1)$ vector, equation (18)
z = complex variable
$z_{(i)}$ = element in row i of $\mathbf{Z}$ vector, equation (18)
$\mathbf{Z}$ = $(N+1)$ vector, equation (18)
q = heat flow
q'' = heat flux
α = solar absorptance
ϵ = emittance
σ = Stefan-Boltzmann constant
ϕ_p = fraction of radiation gains from people, equipment, and lights that is absorbed at the internal surface of building element k
ψ_1 = dimensionless error function, equation (25)
ψ_2 = weighted error function, equation (26)
$\overline{(\tau\alpha)}_k$ = fraction of incident solar radiation that is absorbed at the interior surface of building element k

Subscripts

a = wall a
amb = ambient temperature
b = wall b
i = inside
I = solar radiation
k = building element k
load = heating or cooling load for building
n = transfer function coefficient for n time steps prior to time t
o = outside
r = room air
rad = radiation gains from people equipment and lights
sa = sol-air temperature, equation (30)
$t-n\delta$ = input or output n time steps prior to time t
χ = combined

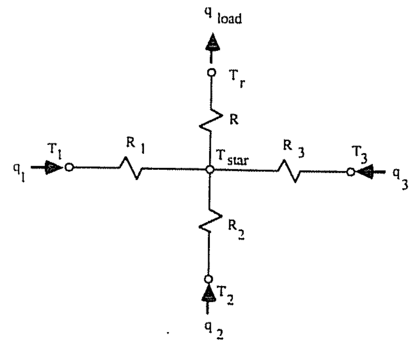

Fig. 2 Star network for a room with three surfaces

The Star Network

Direct combination of individual wall transfer functions into a CRTF when view factors are used to model long-wave radiation in the room requires the manipulation of polynomial matrices (Chen, 1984), an area of research in systems analysis. Approximating the actual radiation and convection heat transfer processes in a room by a star network allows for each combination (i.e., no manipulation of polynomial matrices) of transfer functions. Figure 2 shows a star network for an enclosure with three surfaces. This section presents a computationally easy method for determining the resistances in a star network that uses wall-to-wall view factors to model long-wave radiation exchange, i.e., a view factor network. (Davies (1983) presents a method for modeling the radiation heat transfer processes in a room with a star network. The star network developed in this paper models both the radiation and convection heat transfer processes in a room.)

Figure 3 shows a view factor network for an enclosure with three surfaces. The resistance to long-wave radiation exchange between surfaces in the view factor network is

$$R_{i-j,\text{rad}} = \frac{1}{\epsilon_i A_i G_{i-j} \sigma 4 \bar{T}^3} \tag{15}$$

The absorption factor (Gebhart, 1971), G_{i-j}, is the fraction of energy emitted by surface i that is absorbed by surface j.

Two main steps are involved in determining the resistances in the star network from the resistances in the view factor network. First, the resistance between each pair of nodes in the view factor network is determined when all other nodes are floating. A floating node is defined as one in which heat transfer occurs only by convection to the air or by long-wave radiation exchange with other surfaces in the enclosure. As a result, conduction through building elements, solar radiation gains, and radiation gains from people, equipment, and lights do not affect floating surface nodes, and infiltration or convection gains from people, equipment, and lights do not affect the floating room air node. Second, an approximation is used to determine the resistances in the star network from the resistances between nodes in the view factor network.

To compute the resistances between nodes in the view factor network, an energy balance must be performed on each surface in the enclosure and on the air in the enclosure. An energy balance for surface i in an enclosure with N surfaces is

$$\frac{(T_1 - T_i)}{R_{1-i,\text{rad}}} + \frac{(T_2 - T_i)}{R_{2-i,\text{rad}}} + \ldots + \frac{(T_N - T_i)}{R_{N-i,\text{rad}}} + \frac{(T_r - T_i)}{R_{i,\text{conv}}} + q_i = 0 \tag{16}$$

where q_i is the energy input to surface i other than by convection to the air or long-wave radiation exchange with other surfaces in the room, e.g., absorbed solar energy. An energy balance for the air in the room is

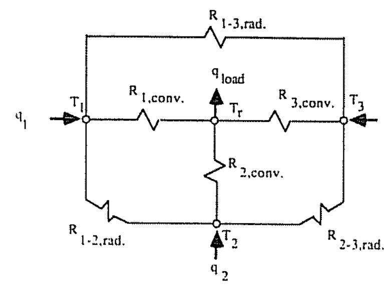

Fig. 3 View factor network for a room with three surfaces

$$\frac{(T_1 - T_r)}{R_{1,\text{conv}}} + \frac{(T_2 - T_r)}{R_{2,\text{conv}}} + \ldots + \frac{(T_N - T_r)}{R_{N,\text{conv}}} = q_{\text{load}} \tag{17}$$

The N energy balances for every surface in the room and the energy balance on the room air can be combined into the matrix equations

$$\mathbf{XY} = \mathbf{Z} \tag{18}$$

where (for $i = 1$ to N and $j = 1$ to N)

$$x_{(i,i)} = -\left(\sum_{j=1}^{N} \frac{1}{R_{i-j,\text{rad}}}\right) - \frac{1}{R_{i,\text{conv}}}$$

$$x_{(i,j)} = x_{(j,i)} = \frac{1}{R_{i-j,\text{rad}}}$$

$$x_{(i,N+1)} = 0$$

$$x_{(N+1,i)} = \frac{1}{R_{i,\text{conv}}}$$

$$x_{(N+1,N+1)} = -1$$

$$y_{(i)} = T_i - T_r$$

$$y_{(N+1)} = q_{\text{load}}$$

$$z_{(i)} = -q_i$$

$$z_{(N+1)} = 0$$

To compute the resistance R_{i-j} between surfaces i and j when all other nodes are floating, arbitrarily let q_i be unit y so that

$$q_i = -q_j = 1.0 \text{ [W or Btu/h]} = \frac{T_i - T_j}{R_{i-j}} \tag{19}$$

then

$$R_{i-j} = (T_i - T_r) - (T_j - T_r) = y_{(i)} - y_{(j)}$$
$$= x_{(i,j),\text{inv}} + x_{(j,i),\text{inv}} - x_{(i,i),\text{inv}} - x_{(j,j),\text{inv}} \tag{20}$$

To compute the resistance R_{i-r} between surface i and the room air, again let

$$q_i = q_{\text{load}} = \frac{(T_i - T_r)}{R_{i-r}} = 1.0 \text{ [W or Btu/h]} \tag{21}$$

then

$$R_{i-r} = T_i - T_r = -x_{(i,i),\text{inv}} \tag{22}$$

A number of approximations could be used to obtain the unknown resistances in the star network from the resistances between nodes in the view factor network. For example, nonlinear regression could be used to minimize the error in heat flow between nodes, or linear regression could be used to minimize the resistance to heat transfer between nodes. An approximation that accurately models the heat transfer processes in a room and requires less computational effort than linear or nonlinear regression is as follows.

The net heat flow to the air (i.e., load) for steady-state temperature differences between enclosure surfaces and the air will be the same for the star network and the view factor network if the following N equations are satisfied:

$$R_1 + R = R_{1-r}$$
$$R_2 + R = R_{2-r}$$
$$\vdots$$
$$R_N + R = R_{N-r} \tag{23}$$

One more equation is needed to provide $N+1$ equations with $N+1$ unknowns. The last equation is selected so that the heat transfer between surfaces for the star network approximates the heat transfer between surfaces in the view factor network. The difference in resistance to heat transfer between surface nodes i and j when other nodes are floating for the view factor network and the star network is

$$R_i + R_j - R_{i-j} = R_{i-r} + R_{j-r} - R_{i-j} - 2R \tag{24}$$

Squaring a dimensionless form of this error in resistance between all surface nodes gives the function

$$\psi_1 = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{(R_{i-r} + R_{j-r} - R_{i-j} - 2R)^2}{R_{i-j}^2} \tag{25}$$

Surfaces with a lower resistance (R_{i-j}) between them have a larger heat transfer for the same temperature difference. The following error function will place more weight on resistances between surfaces with a smaller R_{i-j}:

$$\psi_2 = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{(R_{i-r} + R_{j-r} - R_{i-j} - 2R)^2}{R_{i-j}^3} \tag{26}$$

Other weighting functions could be used to obtain ψ_2, but as will be shown, this weighting function results in accurate modeling of the heat transfer processes for rooms with a wide variety of thermal physical properties, resistances to long-wave radiation exchange, and resistances to convective heat transfer. The derivative of ψ_2 with respect to resistance R between the star node and the air is

$$\frac{d(\psi_2)}{dR} = \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{-4(R_{i-r} + R_{j-r} - R_{i-j} - 2R)}{R_{i-j}^3} \tag{27}$$

Setting the derivative ψ_2 with respect to R equal to zero gives

$$R = \frac{\displaystyle\sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{R_{i-r} + R_{j-r} - R_{i-j}}{R_{i-j}^3}}{\displaystyle 2\sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{1}{R_{i-j}^3}} \tag{28}$$

The second derivative of ψ_2 with respect to R will be positive for all positive values of R. Therefore, ψ_2 will be a minimum when R is positive. The other N unknown resistances in the star network can be obtained from equation (23) after equation (28) is used to compute the resistance between the star node and the room air node. Seem (1987) shows that this method results in an exact transformation for a room with two surfaces and for rooms that have the same resistance to long-wave radiation exchange and resistance to convective heat transfer for all surfaces.

Loads with the star and view factor networks were compared for a three-surface room and an eight-surface room. To test the star network, building elements with a wide range of thermal physical properties, resistances to long-wave radiation exchange, and resistances were used. The three-surface room contained the following building elements:

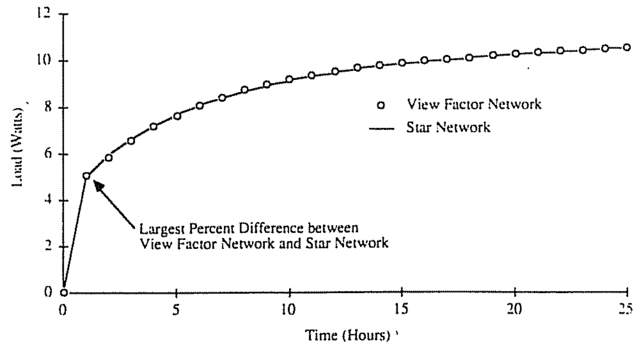

Fig. 4 Response to 0.56°C (1°F) step change in outdoor temperature for eight-surface room

Table 1 Percent difference in steady-state loads between the star and view factor networks

Input	Three-Surface Room	Eight-Surface Room
Temperature Difference	0.45 percent	0.69 percent
Solar	0.12 percent	0.36 percent
Radiation	0.006 percent	0.17 percent

1) 3 m^2 of exterior glazing
2) 3 m^2 of an exterior frame wall with 80 mm of insulation
3) 30 m^2 of 0.3 m heavy concrete interior partition

The eight-surface room contained the following building elements:

1) 6 m^2 of exterior glazing
2) 1 m^2 of the stud path of an exterior frame wall
3) 5 m^2 of the insulation path of an exterior frame wall
4) 12 m^2 of an 0.2 m low-weight concrete floor deck
5) 6 m^2 of a frame partition with 30 mm wood
6) 12 m^2 of interior partition consisting of an 0.2 m clay tile with 20 mm plaster
7) 6 m^2 of interior partition consisting of 0.1 m clay tile with 20 mm plaster
8) 12 m^2 of a 0.1 m wood deck with false ceiling

Loads resulting from step changes in outdoor temperature, indoor temperature, solar radiation gains, and radiation gains from people, equipment, and lights were computed for the star and view factor networks for both rooms. A one-hour time step was used. The time step with the largest percent difference in loads between the networks is shown in Fig 4. Table 1 contains the percent difference in steady-state loads between the star and view factor networks for the following inputs: temperature difference between the ambient and indoor air, solar radiation gains, and radiation gains from people, equipment, and lights. Figure 4 and Table 1 demonstrate that the star network accurately models the heat transfer processes for both rooms and all inputs.

This section described a method for determining the resistances in a star network from a network that uses view factors to model long-wave radiation exchange. Locations of room surfaces are needed to compute view factors. Carroll (1980) developed a mean radiant temperature (MRT) network for modeling the long-wave radiation exchange in rooms. There are two advantages of the MRT network over the view factor network. First, no information concerning the location of room surfaces is required. Second, it is easier to include furnishings in a MRT network (Walton, 1984). The resistances in the star network could also be obtained from a MRT network.

Transfer Function Combination for a Star Network

This section presents a method for combining individual

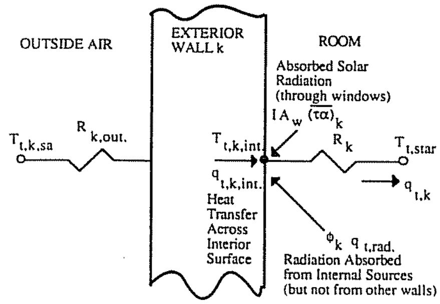

Fig. 5 Energy flows for an exterior wall

building component transfer functions into a single transfer function for an enclosure modeled by a star network. This method requires three main steps. First, transfer functions are developed for each building component (e.g., wall, floor, ceiling) that relate heat flow to the star node with the inputs. Second, transfer functions for each building component are combined in order to relate the total heat flow to the star node with the inputs. Third, the star temperature is removed from the combined transfer function equation by relating the building load to the temperature difference between the star node and the room air temperature.

A transfer function equation for an exterior wall will be developed that will relate the heat flow to the star node with solar radiation gains (from one direction), outdoor temperature, star temperature, and radiation gains from people, equipment, and lights as indicated in Fig. 5. The methods previously discussed can be used to calculate the coefficients for a transfer function equation of the form

$$q_{t,k,\mathrm{int}} = \sum_{n=0} (a_{n,k}A_k T_{t-n\delta,k,sa} + b_{n,k}A_k T_{t-n\delta,k,\mathrm{int}}) - \sum_{n=1} c_{n,k} q_{t-n\delta,k,\mathrm{int}} \tag{29}$$

The sol-air temperature (Mitchell, 1983) for wall k is

$$T_{t-n\delta,k,sa} = T_{t-n\delta,\mathrm{amb}} + I_{t-n\delta}\alpha_k R_{k,\mathrm{out}} A_k \tag{30}$$

An energy balance on the interior surface results in the following equation for heat flow to the star node from wall k:

$$q_{t-n\delta,k,\mathrm{int}} = q_{t-n\delta,k} - \phi_k q_{t-n\delta,\mathrm{rad}} - I_{t-n\delta}A_w \overline{(\tau\alpha)}_k \tag{31}$$

The interior surface temperature is related to the star temperature and heat flow to the star node by

$$T_{t-n\delta,k,\mathrm{int}} = R_k q_{t-n\delta,k} + T_{t-n\delta,\mathrm{star}} \tag{32}$$

Substituting equations (30), (31), and (32) into equation (29) results in

$$q_{t,k} = \sum_{n=0} (d_{n,k}T_{t-n\delta,\mathrm{amb}} + e_{n,k}T_{t-n\delta,\mathrm{star}} + f_{n,k}I_{t-n\delta} + g_{n,k}q_{t-n\delta,\mathrm{rad}}) - \sum_{n=1} h_{n,k}q_{t-n\delta,k} \tag{33}$$

where

$$d_{n,k} = \frac{a_{n,k}A_k}{1-b_{0,k}A_kR_k} \tag{34}$$

$$e_{n,k} = \frac{b_{n,k}A_k}{1-b_{0,k}A_kR_k} \tag{35}$$

$$f_{n,k} = \frac{c_{n,k}A_w\overline{(\tau\alpha)}_k + a_{n,k}\alpha_k R_{k,\mathrm{out}}A_k^2}{1-b_{0,k}A_kR_k} \tag{36}$$

$$g_{n,k} = \frac{c_{n,k}\phi_k}{1-b_{0,k}R_kA_k} \tag{37}$$

$$h_{n,k} = \frac{c_{n,k} - b_{n,k}R_kA_k}{1-b_{0,k}R_kA_k} \tag{38}$$

Equation (33) relates the heat flow to the star node for exterior wall k with the inputs. Seem (1987) develops similar transfer functions for heat flow to the star node from an interior partition and a window. The equations for an interior partition are similar to the equations for an exterior wall, but the surface temperature on both sides of the interior partition are identical. For a window, there are no past time steps involved in the transfer function equation because the thermal capacitance of glass is small.

The transfer functions for heat flow from each building element to the star node can be combined in a nested fashion (i.e., combine the transfer functions for building elements 1 and 2, then combine the transfer function for building element 3 with the combined transfer function for building elements 1 and 2, and continue). The backshift operator can be used to combine any two transfer functions to give the combined heat flow $q_{t,x}$:

$$q_{t,x} = \sum_{n=0} (d_{n,x}T_{t-n\delta,\mathrm{amb}} + e_{n,x}T_{t-n\delta,\mathrm{star}} + f_{n,x}I_{t-n\delta} + g_{n,x}q_{t-n\delta,\mathrm{rad}}) - \sum_{n=1} h_{n,x}q_{t-n\delta,x} \tag{39}$$

where

$$q_{t-n\delta,x} = q_{t-n\delta,1} + q_{t-n\delta,2} \tag{40}$$

$$d_{n,x} = \sum_{j=0}^{n} (d_{n-j,1}h_{j,2} + d_{j,2}h_{n-j,1}) \tag{41}$$

$$e_{n,x} = \sum_{j=0}^{n} (e_{n-j,1}h_{j,2} + e_{j,2}h_{n-j,1}) \tag{42}$$

$$f_{n,x} = \sum_{j=0}^{n} (f_{n-j,1}h_{j,2} + f_{j,2}h_{n-j,1}) \tag{43}$$

$$g_{n,x} = \sum_{j=0}^{n} (g_{n-j,1}h_{j,2} + g_{j,2}h_{n-j,1}) \tag{44}$$

$$h_{n,x} = \sum_{j=0}^{n} (h_{j,1}h_{n-j,2}) \tag{45}$$

Combining heat flows from every surface to the star node gives

$$q_{t,x} = ((\ldots(((q_{t,1}+q_{t,2})+q_{t,3})+q_{t,4}) + \ldots + q_{t,n-1}) + q_{t,n}) = q_{t,\mathrm{load}} \tag{46}$$

The load is related to the temperature difference between the star node and the air node simply by

$$q_{t,\mathrm{load}} = \frac{T_{t,\mathrm{star}} - T_{t,r}}{R} \tag{47}$$

Substituting equations (46) and (47) into equation (39) gives

$$q_{t,\mathrm{load}} = \sum_{n=0} (d_nT_{t-n\delta,\mathrm{amb}} + e_nT_{t-n\delta,r} + f_nI_{t-n\delta} + g_nq_{t-n\delta,\mathrm{rad}}) - \sum_{n=1} h_nq_{t-n\delta,\mathrm{load}} \tag{48}$$

where

$$d_n = \frac{d_{n,x}}{1 - Re_{0,x}} \quad (49)$$

$$e_n = \frac{e_{n,x}}{1 - Re_{0,x}} \quad (50)$$

$$f_n = \frac{f_{n,x}}{1 - Re_{0,x}} \quad (51)$$

$$g_n = \frac{g_{n,x}}{1 - Re_{0,x}} \quad (52)$$

$$h_n = \frac{h_{n,x} - Re_{n,x}}{1 - Re_{0,x}} \quad (53)$$

Equation (48) is a CRTF, which relates the load for an enclosure to past and current inputs and past loads. The next section presents a method for reducing the computational effort of CRTF simulations.

Model Reduction

The multiple input CRTF given by equation (48) involves more coefficients than required for any of the individual building components. Not all of these coefficients are needed, however, and computational effort can be significantly reduced by model reduction, i.e., find a smaller set of coefficients (i.e., requiring less past information) that provides nearly the same results. Model reduction techniques for multiple input CRTF are complex. A simpler approach to model reduction is to use superposition to decompose the multiple input CRTF into single input CRTF's. Performing a simulation with the single input CRTF's would require more computational effort than would be required with the multiple input CRTF [equation (48)], but the model reduction method discussed in this section can be used to obtain a reduced set (i.e., fewer coefficients) of single input CRTF's, which greatly reduces the necessary computational effort.

The following four single input CRTF's are required to compute the same building loads as given by the multiple input CRTF, equation (48):

$$q_{t,\text{load,amb}} = \sum_{n=0} (d_n T_{t-n\delta,\text{amb}}) - \sum_{n=1} (h_n q_{t-n\delta,\text{load,amb}}) \quad (54)$$

$$q_{t,\text{load},r} = \sum_{n=0} (e_n T_{t-n\delta,r}) - \sum_{n=1} (h_n q_{t-n\delta,\text{load},r}) \quad (55)$$

$$q_{t,\text{load},I} = \sum_{n=0} (f_n I_{t-n\delta}) - \sum_{n=1} (h_n q_{t-n\delta,\text{load},I}) \quad (56)$$

$$q_{t,\text{load,rad}} = \sum_{n=0} (g_n q_{t-n\delta,\text{rad}}) - \sum_{n=1} (h_n q_{t-n\delta,\text{load,rad}}) \quad (57)$$

The net heating or cooling load for a room is

$$q_{t,\text{load}} = q_{t,\text{load,amb}} + q_{t,\text{load},r} + q_{t,\text{load},I} + q_{t,\text{load,rad}} \quad (58)$$

Shamash (1980) said that the Padé approximation is a popular method for reducing single-input Laplace transfer functions because it requires little computational effort, cancels common factors if they exist, and matches the steady-state response of the original and reduced Laplace transfer functions for polynomial inputs. This section extends the Padé approximation to model reduction for single-input transfer functions.

The following single-input transfer function relates the inputs of a system to the outputs:

$$y_t = \sum_{j=0}^{n} (a_j u_{t-j\delta}) - \sum_{j=1}^{n} (b_j y_{t-j\delta}) \quad (59)$$

Taking the z transformation (Jury, 1964) of equation (59) results in

$$\left[\sum_{j=0}^{n} (b_j z^{-j})\right] Y(z) = \left[\sum_{j=0}^{n} (a_j z^{-j})\right] U(z) \quad (60)$$

From equation (60), the z transfer function is

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\sum_{j=0}^{n} a_j z^{-j}}{\sum_{j=0}^{n} b_j z^{-j}} \quad (61)$$

Equation (61) is a z transfer function that relates the z transform of the input to the z transform of the output, and it is unstable if there are poles outside the unit circle. (Poles are roots of the denominator of a transfer function.) To reduce z transfer functions, the bilinear transformation (Kuo, 1980)

$$z = \frac{1+w}{1-w} \quad (62)$$

is used to transform a z transfer function into a w transfer function. (A w transfer function is a ratio of polynomials of the complex variable w.) The bilinear transformation maps the unit circle on the z plane into the left half of the w plane. A w transfer function behaves like a Laplace transfer function because both transfer functions are unstable if they have poles in the right half of their complex planes. Substituting equation (62) into equation (61) gives the following w transfer function:

$$G(w) = \frac{\sum_{j=0}^{n} a_j \left(\frac{1+w}{1-w}\right)^{-j}}{\sum_{j=0}^{n} b_j \left(\frac{1+w}{1-w}\right)^{-j}} = \frac{\sum_{j=0}^{n} a_j \left(\frac{1-w}{1+w}\right)^{j}}{\sum_{j=0}^{n} b_j \left(\frac{1-w}{1+w}\right)^{j}}$$

$$= \frac{\sum_{j=0}^{n} a_j (1-w)^j (1+w)^{n-j}}{\sum_{j=0}^{n} b_j (1-w)^j (1+w)^{n-j}} \quad (63)$$

Appendix A describes an algorithm for determing the $\nu_{i(j,n)}$ coefficients in the following equation:

$$(1-w)^j (1+w)^{n-j} = \sum_{i=0}^{n} \nu_{i(j,n)} w^i \quad (64)$$

Substituting equation (64) into equation (63) gives

$$G(w) = \frac{\sum_{j=0}^{n} a_j \left(\sum_{i=0}^{n} \nu_{i(j,n)} w^i\right)}{\sum_{j=0}^{n} b_j \left(\sum_{i=0}^{n} \nu_{i(j,n)} w^i\right)} = \frac{\sum_{i=0}^{n} \left(\sum_{j=0}^{n} a_j \nu_{i(j,n)}\right) w^i}{\sum_{i=0}^{n} \left(\sum_{j=0}^{n} b_j \nu_{i(j,n)}\right) w^i} \quad (65)$$

Equation (65) is a ratio of polynomials of the complex variable w. Appendix B contains equations for reducing Laplace transfer functions with the Padé approximation. The Padé approximation can also be used to reduce w transfer functions. To use the equations in Appendix B to reduce w transfer functions, the coefficient for w^0 in the denominator of equation (65) must be set equal to one.

Table 2 **Transfer function coefficients for exterior wall 17**

	ASHRAE		Reduced	
i	a_i, W/(m²-K)	c_i	a_i, W/(m²-K)	c_i
0	0	1	−0.00505	1
1	0.00006	−2.35214	0.02339	−2.13547
2	0.00125	1.98104	−0.03895	1.47238
3	0.00511	−0.73353	0.03486	−0.32658
4	0.00454	0.12178		
5	0.00108	−0.00859		
6	0.00006	0.00021		

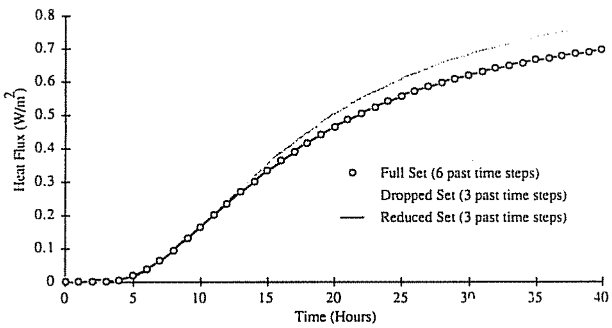

Fig.6 Response to 0.56°C (1°F) step change in outdoor temperature for ASHRAE exterior wall 17

$$G(w)=\frac{\sum_{i=0}^{n}\bar{a}_i w^i}{\sum_{i=0}^{n}\bar{b}_i w^i} \tag{66}$$

where

$$\bar{a}_i=\frac{\sum_{j=0}^{n} a_j \nu_{i(j,n)}}{\sum_{j=0}^{n} b_j \nu_{0(j,n)}} \tag{67}$$

$$\bar{b}_i=\frac{\sum_{j=0}^{n} b_j \nu_{i(j,n)}}{\sum_{j=0}^{n} b_j \nu_{0(j,n)}} \tag{68}$$

The Padé approximation described in Appendix B can be used to obtain a reduced w transfer function of the following form:

$$G_r(w)=\frac{\sum_{i=0}^{m}\bar{d}_i w^i}{\sum_{i=0}^{m}\bar{e}_i w^i} \tag{69}$$

where

$\bar{e}_0=1$

m = number of past time steps in reduced transfer function

Next, the reduced w transfer is transformed into a reduced z transfer function by using the bilinear transformation

$$w=\frac{z-1}{z+1}=\frac{1-z^{-1}}{1+z^{-1}} \tag{70}$$

Substituting equation (70) into equation (69) results in

$$G_r(z)=\frac{\sum_{i=0}^{m}\bar{d}_i\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^i}{\sum_{i=0}^{m}\bar{e}_i\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^i}=\frac{\sum_{i=0}^{m}\bar{d}_i(1-z^{-1})^i(1+z^{-1})^{m-i}}{\sum_{i=0}^{m}\bar{e}_i(1-z^{-1})^i(1+z^{-1})^{m-i}} \tag{71}$$

The algorithm described in Appendix A can be used to compute the $\nu_{j(i,m)}$ coefficients in the following equation:

$$(1-z^{-1})^i(1+z^{-1})^{m-i}=\sum_{j=0}^{m}\nu_{j(i,m)}z^{-j} \tag{72}$$

Substituting equation (72) into equation (71) gives

$$G_r(z)=\frac{\sum_{i=0}^{m}\bar{d}_i\left(\sum_{j=0}^{m}\nu_{j(i,m)}z^{-j}\right)}{\sum_{i=0}^{m}\bar{e}_i\left(\sum_{j=0}^{m}\nu_{j(i,m)}z^{-j}\right)}=\frac{\sum_{j=0}^{m}\left(\sum_{i=0}^{m}\bar{d}_i\nu_{j(i,m)}\right)z^{-j}}{\sum_{j=0}^{m}\left(\sum_{i=0}^{m}\bar{e}_i\nu_{j(i,m)}\right)z^{-j}} \tag{73}$$

Equation (73) can be rewritten as

$$G_r(z)=\frac{\sum_{j=0}^{m}d_j z^{-j}}{\sum_{j=0}^{m}e_j z^{-j}} \tag{74}$$

where

$$d_j=\frac{\sum_{i=0}^{m}\bar{d}_i\nu_{j(i,m)}}{\sum_{i=0}^{m}\bar{e}_i\nu_{0(i,m)}} \tag{75}$$

$$e_j=\frac{\sum_{i=0}^{m}\bar{e}_i\nu_{j(i,m)}}{\sum_{i=0}^{m}\bar{e}_i\nu_{0(i,m)}} \tag{76}$$

Transforming equation (76) back into the time domain gives the following reduced transfer function:

$$y_t=\sum_{j=0}^{m}d_j u_{t-j\delta}-\sum_{j=1}^{m}e_j y_{t-j\delta} \tag{77}$$

ASHRAE contains tables of single input transfer functions for walls, roofs, and interior partitions. The Padé approximation and bilinear tranformation can be used to obtain a reduced set of coefficients that closely model the response of the full set of coefficients listed in ASHRAE. As an example, Table 2 contains transfer function coefficients for exterior wall 17 (4-in. face brick, 8-in. common brick with air space) listed in ASHRAE. Table 2 also contains reduced transfer function coefficients for three time steps back rather than six as given by ASHRAE. Figure 6 is a plot of the response to a 0.56°C (1°F) step input for the full set of coefficients, the reduced set of coefficients, and a dropped set of the coefficients, i.e., the ASHRAE coefficients for three time steps back. The reduced coefficients closely reproduce the full set of ASHRAE coefficients while the dropped set results in errors.

When combining transfer functions for building elements, the number of past time steps in the resulting transfer function increases. Fortunately, the number of past time steps required to perform a simulation accurately can be significantly reduced by using the Padé approximation and bilinear transformation. Figure 7 shows the response to a 0.56°C (1°F) step change in outdoor temperature for full and reduced sets of single input CRTF's for the eight-surface room.

Comparison of Methods

Table 3 contains the number of mutiplications required per time step for energy balance simulations of view factor net-

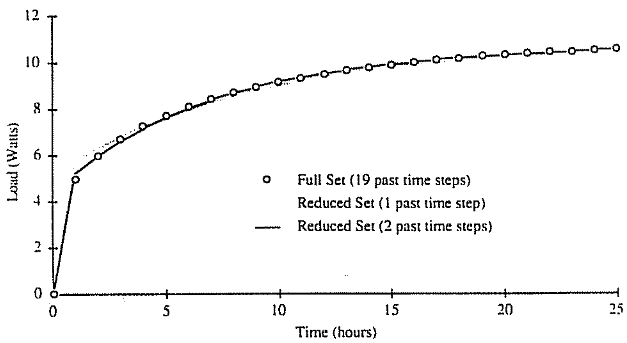

Fig.7 Response to 0.56°C (1°F) step change in outdoor temperature for eight-surface room

works with time-independent **A** matrices, multiple-input CRTF simulations, and single-input CRTF's simulations with reduced coefficients for the rooms previously described. For the methods compared in Table 3, the number of additions required per time step is close to the number of multiplications required per time step. Table 3 demonstrates that the computational savings of using reduced single-input CRTF's increase with the complexity of the zone, i.e., the larger the number of surfaces the greater the computational savings.

Table 3 shows that the computational effort of single-input CRTF simulations with reduced coefficients is less than the computational effort of energy balance simulations with time-independent **A** matrices. Sowell and Walton (1980) determined that the execution times are similar for energy balance simulations with time-dependent **A** matrices and DOE 2.1 (1980). (DOE 2.1 is an advanced weighting factor program, which assumes that heat transfer processes are linear and time-invariant.) The computational effort of energy balance simulations with time-independent **A** matrices is less than the computational effort of energy balance simulations with time-dependent **A** matrices. Therefore, reduced CRTF simulations should require significantly less effort than DOE 2.1 simulations.

Conclusion

A procedure for accurately and efficiently computing loads and floating room temperatures in buildings is presented in this paper. Three main steps are involved in this procedure. First, the resistances for a star network are computed from the resistances for a view factor network. Second, transfer functions for individual building elements of a star network are combined. Third, the Padé approximation and bilinear transformation are used to reduce the number of coefficients in the combined transfer function equation.

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Table 3 Number of multiplications required per time step

	Energy balance	Multiple-input CRTF	Reduced single-input CRTF's
Three-surface room	35	38	20
Eight-surface room	152	109	20

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APPENDIX A

Extension of Pascal's Triangle

Numerical analysis textbooks (Conte and de Boor, 1980; Sedgewick, 1983) contain algorithms for multiplying polynomials. These algorithms could be used to obtain the $\nu_{j(i,n)}$ coefficients in the following equation:

$$(1-x)^i(1+x)^{n-i} = \sum_{j=0}^{n} \nu_{j(i,n)} x^j \tag{78}$$

This appendix contains a numerically efficient algorithm for determining the $\nu_{j(i,n)}$ coefficients in equation (78). The algorithm is numerically efficient because no multiplications or divisions are required. The following algorithm for computing the coefficients in equation (78) is based upon an extension of Pascal's triangle (Spiegel, 1968):

$\nu_{0(i,n)} = 1$

For $k=1$ to i with a step size of 1

$\nu_{k(i,n)} = -\nu_{k-1(i,n)}$

For $j=(k-1)$ to 1 with a step size of -1

$\nu_{j(i,n)} = \nu_{j(i,n)} - \nu_{j-1(i,n)}$

Next j

Next k

For $k=(i+1)$ to n with a step size of 1

$\nu_{k(i,n)} = \nu_{k-1(i,n)}$

For $j=(k-1)$ to 1 with a step size of -1

$\nu_{j(i,n)} = \nu_{j(i,n)} + \nu_{j-1(i,n)}$

Next j

Next k

APPENDIX B

Padé Approximation

Jamshidi (1983) presented equations for reducing Laplace (i.e., continuous) transfer functions when the order of the numerator is equal to or less than the order of the denominator. In this paper, equations for reducing Laplace transfer functions when the order of the numerator is equal to the order of the denominator are needed. Therefore, this appendix contains equations for reducing single-input Laplace transfer functions with the Padé approximation when the order of the numerator is equal to the order of the denominator.

The reduced Laplace transfer function

$$G_r(s) = \frac{\bar{d}_0 + \bar{d}_1 s + \bar{d}_2 s^2 + \ldots + \bar{d}_m s^m}{1 + \bar{e}_1 s + \bar{e}_2 s^2 + \ldots + \bar{e}_m s^m} \tag{79}$$

is the Padé approximation of

$$G(s) = \frac{\bar{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \ldots + \bar{a}_n s^n}{1 + \bar{b}_1 s + \bar{b}_2 s^2 + \ldots + \bar{b}_n s^n} \tag{80}$$

(where m is less than n) if the power series expansion for $G_r(s)$ is equal to the power series expansion of $G(s)$ for terms of order s^0 to s^{2m}. Next, the equations for calculating the power series of

$$G(s) = \bar{c}_0 + \bar{c}_1 s + \bar{c}_2 s^2 + \bar{c}_3 s^3 + \ldots \tag{81}$$

will be formulated. The following equation results from equating equation (81) with equation (80):

$$\bar{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \ldots + \bar{a}_n s^n = (1 + \bar{b}_1 s + \bar{b}_2 s^2 + \ldots + \bar{b}_n s^n)(\bar{c}_0 + \bar{c}_1 s + \bar{c}_2 s^2 + \ldots \tag{82}$$

Multiplying the terms on the right-hand side of equation (82) together and combining common powers of the Laplace transform variable s results in

$$\bar{a}_0 + \bar{a}_1 s + \bar{a}_2 s^2 + \ldots + \bar{a}_n s^n = \bar{c}_0 + (\bar{c}_0 \bar{b}_1 + \bar{c}_1)s + (\bar{c}_0 \bar{b}_2 + \bar{c}_1 \bar{b}_1 + \bar{c}_2)s^2 + \ldots \tag{83}$$

The $\bar{c}_i$ coefficients for the power series expansion of $G(s)$ are determined by equating the coefficients of equal powers of s in equation (83).

$$\begin{aligned} \bar{c}_0 &= \bar{a}_0 \\ \bar{c}_1 &= \bar{a}_1 - \bar{c}_0 \bar{b}_1 \\ &\ \ \vdots \\ \bar{c}_2 &= \bar{a}_2 - \bar{c}_0 \bar{b}_2 - \bar{c}_1 \bar{b}_1 \\ \bar{c}_i &= \bar{a}_i - \sum_{j=0}^{i-1} \bar{c}_j \bar{b}_{i-j} \end{aligned} \tag{84}$$

To calculate the Padé approximation for $G(s)$, the power series for $G(s)$ is set equal to the Laplace transfer function $G_r(s)$. The following equation results from combining equal powers of s when the numerator of $G_r(s)$ is set equal to the denominator of $G_r(s)$ times $G(s)$:

$$\begin{aligned} \bar{d}_0 + \bar{d}_1 s + \bar{d}_2 s^2 + \ldots + \bar{d}_m s^m &= \bar{c}_0 + (\bar{c}_1 + \bar{e}_1 \bar{c}_0)s \\ &+ (\bar{c}_2 + \bar{e}_1 \bar{c}_1 + \bar{e}_2 \bar{c}_0)s^2 + \ldots \\ &+ (\bar{c}_m + \bar{e}_1 \bar{c}_{m-1} + \bar{e}_2 \bar{c}_{m-2} + \ldots + \bar{e}_m \bar{c}_0)s^m \\ &+ (\bar{c}_{m+1} + \bar{e}_1 \bar{c}_m + \bar{e}_2 \bar{c}_{m-1} + \ldots + \bar{e}_m \bar{c}_1)s^{m+1} \\ &+ (\bar{c}_{m+2} + \bar{e}_1 \bar{c}_{m+1} + \bar{e}_2 \bar{c}_m + \ldots + \bar{e}_m \bar{c}_2)s^{m+2} + \ldots \\ &+ (\bar{c}_{2m} + \bar{e}_1 \bar{c}_{2m-1} + \bar{e}_2 \bar{c}_{2m-2} + \ldots + \bar{e}_m \bar{c}_m)s^{2m} + \ldots \end{aligned} \tag{85}$$

A set of m linear equations with m unknown denominator coefficients ($\bar{e}_i$) of the Padé approximation can be formulated by equating powers of s from $(m+1)$ to $(2m)$ in equation (85).

$$\begin{bmatrix} \bar{c}_m & \bar{c}_{m-1} & \ldots & \bar{c}_2 & \bar{c}_1 \\ \bar{c}_{m+1} & \bar{c}_m & \ldots & \bar{c}_3 & \bar{c}_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{c}_{2m-2} & \bar{c}_{2m-3} & \ldots & \bar{c}_m & \bar{c}_{m-1} \\ \bar{c}_{2m-1} & \bar{c}_{2m-2} & \ldots & \bar{c}_{m+1} & \bar{c}_m \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{e}_{m-1} \\ \bar{e}_m \end{bmatrix} = \begin{bmatrix} -\bar{c}_{m+1} \\ -\bar{c}_{m+2} \\ \cdot \\ \cdot \\ \cdot \\ -\bar{c}_{2m-1} \\ -\bar{c}_{2m} \end{bmatrix} \tag{86}$$

The denominator of the Padé approximation is determined by solving equation (86) for the $e_1, \bar{e}_2, \ldots, \bar{e}_m$ coefficients. After determining the denominator of the Padé approximation, numerator coefficients of the Padé approximation are determined by equating powers of s from 0 to m in equation (85).

$$\begin{aligned} \bar{d}_0 &= \bar{c}_0 \\ \bar{d}_1 &= \bar{c}_1 + \bar{e}_1 \bar{c}_0 \\ \bar{d}_2 &= \bar{c}_2 + \bar{e}_1 \bar{c}_1 + \bar{e}_2 \bar{c}_0 \\ &\ \ \vdots \\ \bar{d}_m &= \bar{c}_m + \sum_{i=1}^{m} \bar{e}_i \bar{c}_{m-i} \end{aligned} \tag{87}$$

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Techniques for in Situ Determination of Thermal Resistance of Lightweight Board Insulations

Four techniques for determining the in situ thermal resistance of rigid board insulation installed in conventional low-sloped roofs are described and compared. All techniques use measured temperature distributions and heat fluxes in the roof systems. The limitations of the techniques are discussed. Test results are presented to allow a comparison of the methods.

1 Introduction

Thermal resistances (R-values) of roofs or other building envelope systems are normally obtained by a calculation based on the conductivity of the individual materials making up the components. For a one-dimensional layered geometry, e.g., an insulated low-slope roof, R is given by

$$R = \sum_i t_i/k_i \tag{1}$$

where t_i and k_i are, respectively, the thickness and thermal conductivity of the ith layer. For more complex geometries, i.e., multiple heat flow paths, other methods such as the ASHRAE Zone Method (*ASHRAE Handbook*, 1985) can be used. Frequently, these values are confirmed by laboratory hot box experiments (Mumaw, 1980; Van Geem, 1984) that can simulate the geometry of roof and wall components. The actual installed R-value of a system may differ significantly from a calculated or laboratory value due to material differences, aging, thermal bridging, moisture, the quality of construction, and other factors. Field measurement is the only unambiguous way to determine the in-place thermal resistance. However, there currently is no established, reliable technique for making field measurements of installed R-values of building envelope systems. The purpose of this paper is to describe and discuss several techniques for applicability of determining R-values from field data.

The motivation for this study arises from a need to provide measurements of heat transfer in building envelope systems under field conditions. These measurements are needed to assess the thermal performance of "as-built" envelope systems and to monitor the thermal properties of in-service systems that may change because of time or because of environmental conditions.

Two techniques based on averaging are discussed in Section 2, and two techniques based on the use of least squares are given in Sections 3 and 4. Section 5 provides experimental results and Section 6 gives the conclusions.

All techniques for determining the thermal resistance of field-mounted building insulation specimens require perivelic measurements of the temperature difference and the heat flow across the specimen over an extended period of time. Thermocouples are placed on either side of the specimen and a heat flux transducer is mounted within or at the boundary of the specimen. Sensor calibration is critical. Thermocouples cut from the same wire spools are preferred and ASTM procedures should be followed for the heat flux transducer (ASTM C1046-85).

2 Averaging Techniques for Determination of Thermal Resistance From in Situ Data

2.1 Averaging Technique. One method for determining the R-value of a building system in the field is the Averaging Technique (Flanders, 1980), which gives the ratio of the average temperature difference across a sample divided by the average heat flux. This expression for R-value can be derived from the one-dimensional transient heat conduction equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial I}{\partial x}\right) = \rho c \frac{\partial T}{\partial t} \tag{2}$$

to show more clearly the limitations of the technique. In equation (2) x is the space coordinate, t is time, T is temperature, k is thermal conductivity, ρ is density, and c is specific heat. Use of equation (2) implies that the heat transfer is by conduction only (no radiation or convection). Integrating equation (2) over a layer of thickness L, from $x = 0$ to $x = L$, gives

$$k\frac{\partial T}{\partial x}\bigg|_{x=L} - k\frac{\partial T}{\partial x}\bigg|_{x=0} = \int_{x=0}^{L_{\rho c}} \rho c \frac{\partial T}{\partial t}\, dx \tag{3}$$

If this expression is then integrated over time for cyclic conditions, which somewhat approximates diurnal temperature variations, one obtains

$$\int_0^t k\frac{\partial T}{\partial x}\bigg|_{x=L} dt - \int_0^t k\frac{\partial T}{\partial x}\bigg|_{x=0} dt = 0 \tag{4}$$

The right side of equation (4) is zero because the integrand is the change in internal energy of the system, which, over one cycle, does not change. Since this result is true for a layer of any thickness, equation (4) can be rewritten to show that, under the conditions of the derivation, the average heat flux at any location in the system is a constant

$$\bar{q} = \frac{1}{t}\int_0^t q(t)dt = -\frac{1}{t}\int_0^t k\frac{\partial T}{\partial x}\, dt \tag{5}$$

Notice the similarity of equation (5) to the steady-state equation

$$q = -k\frac{\partial T}{\partial x} \tag{6}$$

In the Averaging Technique method, k in equation (5) is as-

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