

Model Reduction of Transfer Functions Using a Dominant Root Method

J. E. Seem

Johnson Controls, Inc.,
Milwaukee, WI 53201

S. A. Klein

W. A. Beckman

J. W. Mitchell

Solar Energy Laboratory,
University of Wisconsin—Madison,
Madison, WI 53706

Transfer function methods are more efficient for solving long-time transient heat transfer problems than Euler, Crank-Nicolson, or other classical techniques. Transfer functions relate the output of a linear, time-invariant system to a time series of current and past inputs, and past outputs. Inputs are modeled by a continuous, piecewise linear curve. The computational effort required to perform a simulation with transfer functions can be significantly decreased by using the Padé approximation and bilinear transformation to determine transfer functions with fewer coefficients. This paper presents a new model reduction method for reducing the number of coefficients in transfer functions that are used to solve heat transfer problems. There are two advantages of this method over the Padé approximation and bilinear transformation. First, if the original transfer function is stable, then the reduced transfer function will also be stable. Second, reduced multiple-input single-output transfer functions can be determined by this method.

Introduction

Year-long simulations of the hourly (or shorter time period) heating and cooling loads for buildings are important for sizing heating, ventilating, and air conditioning equipment, determining the effect of a design change or retrofit on energy usage, and developing optimal control strategies. Yearly simulations require a large amount of computational effort because the solution to a long-time transient heat transfer problem must be determined. For long-time solutions, transfer function methods are more efficient than Euler, Crank-Nicolson, or other classical techniques because there is no critical time step and the internal temperature distribution is not calculated. Transfer functions relate the output of a linear, time-invariant system to a time series of current and past inputs, and past outputs. Inputs are modeled by a continuous, piecewise linear curve.

The definition of transfer function used in the field of heat transfer in buildings is different from that used in the field of automatic controls. In automatic controls a transfer function is the Laplace or z-transform of the output divided by the Laplace or z-transform of the input. In heat transfer, a transfer function is a recursive difference equation that relates the output of a linear, time-invariant system to a time series of current and past inputs, and a time series of past outputs. In this paper, the latter definition will be used. Also, this paper uses the Laplace transfer function as the definition for the Laplace transform of the output divided by the Laplace transform of the input and the z-transfer function as the z-transform of the output divided by the z-transform of the input.

Transfer functions for computing heat flow through building elements (e.g., walls, floors, roofs, partitions) are of the form

$$q_i^n = \sum_{j=0}^n (a_j T_{i-j,0} + b_j T_{i-j,i}) - \sum_{j=1}^n (c_j q_{i-j}^n) \quad (1)$$

There are a number of methods available for calculating the transfer function coefficients in equation (1). Stephenson and Mitalas (1971) presented a method for determining transfer functions for one-dimensional heat transfer through multilay-

ered slabs by solving the conduction equation with Laplace and z-transform theory. Hittle (1981) presented a very detailed description of the development of transfer functions for multilayered slabs. Celyan and Myers (1980) and Seem et al. (1989a) presented methods for determining transfer functions for multidimensional heat transfer by determining the exact solution to a system of ordinary differential equations.

Seem et al. (1988b) presented a method in which the transfer functions describing heat flows in building elements can be combined into a single transfer functions for an enclosure, referred to as a comprehensive room transfer function. The number of past time steps in the combined transfer function is equal to the summation of the number of past time steps for the individual transfer functions. Thus, the computational effort of performing a simulation with a comprehensive room transfer function is not significantly different from the effort required to perform a simulation with the individual transfer functions. Fortunately, model reduction methods can be used to reduce the number of significant coefficients in transfer functions.

A number of different model reduction methods have been developed by researchers in the fields of automatic controls and systems analysis. The motivation behind the development of these methods is to reduce computer time for system simulation and to make control system design and analysis easier. Shamash (1980) notes that the Padé approximation is a popular method for reducing single-input Laplace transfer functions because it requires little computational effort, cancels common factors if they exist, and matches the steady-state response of the original and reduced Laplace transfer functions for polynomial inputs. The Padé approximation requires the power series expansion of the original Laplace-transfer function to be equal to the power series expansion of the reduced Laplace-transfer function for terms of order s^0 to s^{2m} . (The number of coefficients in the reduced Laplace-transfer function is equal to $2m + 1$.) There are two disadvantages of the Padé approximation. First, if the original transfer function is stable, then the reduced transfer function is not guaranteed to be stable. Second, reduced multiple-input transfer functions cannot be obtained with the Padé approximation.

Seem et al. (1989b) have used the bilinear transformation and Padé approximation to reduce single-input transfer func-

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$$Y(z) = \frac{\sum_{j=0}^n (a_j z^{n+1-j})}{\prod_{j=0}^n (z - \lambda_j)} \quad (6)$$

where $\lambda_0 = 1$ and $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ are the roots of $z^n + b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_{n-1} z + b_n = 0$ (7)

Prandt and Wu (1983) state that a transfer function will be stable if the absolute value of every root in equation (7) is less than one. For heat transfer problems that have at least one convective or specified temperature boundary condition, the roots of equation (7) are real and between zero and one (Ceylan, 1979). Hittle (1981) shows that all the roots are distinct for partial differential equations describing heat transfer. When all roots are distinct, the response (e.g., temperature or heat flux) to a step change in an input (e.g., temperature or heat flux) is a summation of exponentials. Transfer functions generated from finite-difference/element models or from a combination of transfer functions for building elements may have multiple real roots. Seem (1987) describes a method for eliminating multiple roots in transfer functions.

When the roots are real and distinct, partial fraction expansion can be used to write the z -transform of the output as

$$Y(z) = \frac{\beta_0 z}{z - \lambda_0} + \frac{\beta_1 z}{z - \lambda_1} + \frac{\beta_2 z}{z - \lambda_2} + \dots + \frac{\beta_n z}{z - \lambda_n} \quad (8)$$

where

$$\beta_j = \frac{\sum_{i=0}^n (a_i \lambda_j^{n-i})}{\prod_{i \neq j} (\lambda_j - \lambda_i)} \quad (9)$$

Transforming equation (8) back to the time domain gives

$$y_{t+k\delta} = \beta_0 + \beta_1 \lambda_1^k + \beta_2 \lambda_2^k + \dots + \beta_n \lambda_n^k \quad (10)$$

where $k \geq 0$.

Equation (10) is the explicit solution for the response at time $k\delta$ to a step input at time zero. The response to a step input can be split into two parts: the steady-state response and the transient response. The steady-state response is β_0 and the transient response is

$$y_{t+k\delta} - \beta_0 = \beta_1 \lambda_1^k + \beta_2 \lambda_2^k + \dots + \beta_n \lambda_n^k \quad (11)$$

The summation of the transient response from time zero to infinity is

$$\sum_{k=0}^{\infty} (y_{t+k\delta} - \beta_0) = \sum_{k=0}^{\infty} (\beta_1 \lambda_1^k + \beta_2 \lambda_2^k + \dots + \beta_n \lambda_n^k) = \frac{\beta_1}{1 - \lambda_1} + \frac{\beta_2}{1 - \lambda_2} + \dots + \frac{\beta_n}{1 - \lambda_n} \quad (12)$$

The roots with the largest effect on the transient response are the roots with the largest value of the following quantity:

$$\omega_j = \left| \frac{\beta_j}{1 - \lambda_j} \right| \quad (13)$$

The dominant roots are defined as the roots with the largest effect on the transient response, i.e., the roots with the largest ω_j computed from equation (13). (The third section of this paper shows that the largest root is not always the dominant root.)

Output Coefficients of the Reduced Transfer Function (Step Two). Equations (9) and (13) can be used to determine the m dominant roots of the original transfer function. This section

contains equations for determining the output coefficients from the m dominant roots.

The following equation is used to determine the numerator of the reduced z -transfer function from the m dominant roots:

$$G_r(z) = \frac{z^m \sum_{j=0}^m (d_j z^{-j})}{\prod_{j=1}^m (z - \bar{\lambda}_j)} = \frac{\sum_{j=0}^m (d_j z^{-j})}{\prod_{j=1}^m (1 - \bar{\lambda}_j z^{-1})} \quad (14)$$

where $\bar{\lambda}_j =$ one of the m dominant roots; $m =$ number of past time steps in the reduced transfer function.

Multiplying the m terms together in equation (14) results in

$$G_r(z) = \frac{\sum_{j=0}^m (d_j z^{-j})}{\sum_{j=0}^m (e_j z^{-j})} \quad (15)$$

where

$$e_0 = 1$$

$$e_1 = - \sum_{i=1}^m \bar{\lambda}_i$$

$$e_2 = \sum_{i=1}^{m-1} \left(\bar{\lambda}_i \sum_{j=i+1}^m \bar{\lambda}_j \right)$$

$$e_3 = - \sum_{i=1}^{m-2} \left[\bar{\lambda}_i \sum_{j=i+1}^{m-1} \left(\bar{\lambda}_j \sum_{k=j+1}^m \bar{\lambda}_k \right) \right]$$

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$$e_m = (-1)^m \prod_{i=1}^m \bar{\lambda}_i \quad (16)$$

The e_j coefficients in the reduced z -transfer function, equation (15) are the output coefficients for the reduced transfer function. As an alternative to equation (16), Appendix A contains an algorithm for determining the e_j coefficients in equation (15).

This step used a subset of the roots of the original transfer function (i.e., the dominant roots) to determine the output coefficients of the reduced transfer function. Thus, if all the roots of the original transfer function are less than one in absolute value, then all the roots of the reduced transfer function will be less than one in absolute value. Recall that the transfer function is stable if all the roots are less than one in absolute value. Therefore, if the original transfer function is stable, the reduced transfer function will also be stable.

Original w -Transfer Function (Step Three). The input coefficients of the reduced transfer function are determined by equating the power series expansion of the original w -transfer function with the reduced transfer function for terms of w^0 to w^m . Step four contains equations for determining the power series expansion of the original w transfer function from the original w -transfer function. This step contains equations for determining the original w -transfer function from the original transfer function, i.e., equation (2).

To determine the original w -transfer function, $G(w)$, the bilinear transformation (Kuo, 1980)

$$z = \frac{1+w}{1-w} \quad (17)$$

is substituted into the original z -transfer function, i.e., equation (3). This results in

$$G(w) = \frac{\sum_{j=0}^n a_j \left(\frac{1-w}{1+w}\right)^{-j}}{\sum_{j=0}^n b_j \left(\frac{1-w}{1+w}\right)^{-j}} = \frac{\sum_{j=0}^n a_j (1-w)^j (1+w)^{n-j}}{\sum_{j=0}^n b_j (1-w)^j (1+w)^{n-j}} \quad (18)$$

Appendix B contains an algorithm for determining the $v_{i(j,n)}$ coefficients in the following equation:

$$(1-w)^j (1+w)^{n-j} = \sum_{i=0}^n (v_{i(j,n)} w^i) \quad (19)$$

Substituting equation (19) into equation (18) gives

$$G(w) = \frac{\sum_{j=0}^n a_j \left(\sum_{i=0}^n v_{i(j,n)} w^i \right)}{\sum_{j=0}^n b_j \left(\sum_{i=0}^n v_{i(j,n)} w^i \right)} = \frac{\sum_{i=0}^n \left(\sum_{j=0}^n a_j v_{i(j,n)} \right) w^i}{\sum_{i=0}^n \left(\sum_{j=0}^n b_j v_{i(j,n)} \right) w^i} \quad (20)$$

Rearranging equation (20) results in

$$G(w) = \frac{\sum_{i=0}^n \bar{a}_i w^i}{\sum_{i=0}^n \bar{b}_i w^i} \quad (21)$$

where

$$\bar{a}_i = a_j v_{i(j,n)} \quad (22)$$

$$\bar{b}_i = b_j v_{i(j,n)} \quad (23)$$

Power Series Expansion for the w -Transfer Function (Step Four). Following is a description of equations for determining the power series expansion of the original w -transfer function from the original w -transfer function. The power series expansion for the original w -transfer function is of the form

$$G(w) = \bar{c}_0 + \bar{c}_1 w + \bar{c}_2 w^2 + \bar{c}_3 w^3 + \dots \quad (24)$$

The following equation results from equating equation (24) with equation (20):

$$\bar{a}_0 + \bar{a}_1 w + \dots + \bar{a}_n w^n = (\bar{b}_0 + \bar{b}_1 w + \dots + \bar{b}_n w^n) (\bar{c}_0 + \bar{c}_1 w + \bar{c}_2 w^2 + \dots) \quad (25)$$

Multiplying the terms on the right-hand side of equation (25) together and combining common powers of the complex variable w results in

$$\bar{a}_0 + \bar{a}_1 w + \dots + \bar{a}_n w^n = \bar{b}_0 \bar{c}_0 + (\bar{b}_0 \bar{c}_1 + \bar{b}_1 \bar{c}_0) w + (\bar{b}_0 \bar{c}_2 + \bar{b}_1 \bar{c}_1 + \bar{b}_2 \bar{c}_0) w^2 + \dots \quad (26)$$

The coefficients for the power series expansion of $G(w)$ are determined by equating the coefficients for equal powers of w in equation (26).

$$\bar{c}_0 = \frac{\bar{a}_0}{\bar{b}_0}$$

$$\bar{c}_1 = \frac{\bar{a}_1 - \bar{b}_1 \bar{c}_0}{\bar{b}_0}$$

$$\bar{c}_2 = \frac{\bar{a}_2 - \bar{b}_1 \bar{c}_1 - \bar{b}_2 \bar{c}_0}{\bar{b}_0}$$

$$\bar{c}_m = \frac{\bar{a}_m - \sum_{j=0}^{m-1} \bar{c}_j \bar{b}_{m-j}}{\bar{b}_0} \quad (27)$$

Denominator of the Reduced w -Transfer Function (Step Five). This step contains equations for determining the denominator of the reduced w -transfer function from the e_j coefficients in the reduced z -transfer function, i.e., equation (15). Step six contains the equation for determining the numerator of the reduced w -transfer function from the denominator of the original w -transfer function.

Substituting the bilinear transformation, equation (17), into equation (15) results in the following reduced w -transfer function:

$$G_r(w) = \frac{N_r(w)}{\sum_{j=0}^m \left[e_j \left(\frac{1-w}{1+w} \right)^j \right]} = \frac{[N_r(w)](1+w)^m}{\sum_{j=0}^m [e_j (1-w)^j (1+w)^{m-j}]} \quad (28)$$

where

$N_r(w)$ = function of complex variable w

The algorithm described in appendix B can be used to determine the $v_{i(j,m)}$ coefficients in the following equation:

$$(1-w)^j (1+w)^{m-j} = \sum_{i=0}^m (v_{i(j,m)} w^i) \quad (29)$$

Substituting equation (29) into equation (28) gives

$$G_r(w) = \frac{[N_r(w)](1+w)^m}{\sum_{j=0}^m \left[e_j \sum_{i=0}^m (v_{i(j,m)} w^i) \right]} = \frac{[N_r(w)](1+w)^m}{\sum_{i=0}^m \left[\left(\sum_{j=0}^m e_j v_{i(j,m)} \right) w^i \right]} \quad (30)$$

Rearranging equation (30) gives

$$G_r(w) = \frac{\sum_{i=0}^m \bar{d}_i w^i}{\sum_{i=0}^m \bar{e}_i w^i} \quad (31)$$

where

\bar{d}_i = coefficients to be determined in step six

$$\bar{e}_i = \sum_{j=0}^m e_j v_{i(j,m)} \quad (32)$$

Numerator of Reduced w -Transfer Function (Step Six). This step describes equations for computing the numerator of the reduced w -transfer function, $G_r(w)$, from the

denominator of the reduced w -transfer function and the power series expansion of the original w -transfer function. Equating the reduced w -transfer function, equation (31), with the power series expansion of the original w -transfer function, equation (24), gives

$$\begin{aligned} \bar{d}_0 + \bar{d}_1 w + \bar{d}_2 w^2 + \dots + \bar{d}_m w^m \\ = (\bar{e}_0 + \bar{e}_1 w + \bar{e}_2 w^2 + \dots + \bar{e}_m w^m) (\bar{c}_0 + \bar{c}_1 w + \bar{c}_2 w^2 + \dots) \\ = \bar{e}_0 \bar{c}_0 + (\bar{e}_0 \bar{c}_1 + \bar{e}_1 \bar{c}_0) w + (\bar{e}_0 \bar{c}_2 + \bar{e}_1 \bar{c}_1 + \bar{e}_2 \bar{c}_0) w^2 + \dots \\ + (\bar{e}_0 \bar{c}_m + \bar{e}_1 \bar{c}_{m-1} + \bar{e}_2 \bar{c}_{m-2} + \dots + \bar{e}_m \bar{c}_0) w^m + \dots \quad (33) \end{aligned}$$

The numerator coefficients $\bar{c}_0, \bar{c}_1, \bar{c}_2, \dots, \bar{c}_m$ of the reduced w -transfer function are determined by equating powers of w from zero to m in equation (33).

$$\bar{d}_0 = \bar{e}_0 \bar{c}_0$$

$$\bar{d}_1 = \bar{e}_0 \bar{c}_1 + \bar{e}_1 \bar{c}_0$$

$$\bar{d}_2 = \bar{e}_0 \bar{c}_2 + \bar{e}_1 \bar{c}_1 + \bar{e}_2 \bar{c}_0$$

$$\bar{d}_m = \bar{e}_0 \bar{c}_m + \sum_{i=1}^m \bar{e}_i \bar{c}_{m-i} \quad (34)$$

Input Coefficients From the Reduced Transfer Function (Step Seven). Next, the bilinear transformation and z -transform theory will be used to determine the input coefficients of the reduced transfer function from the reduced w -transfer function. The reduced w -transfer function, $G_r(w)$, can be transformed into a reduced z -transfer function by using the bilinear transformation

$$w = \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}} \quad (35)$$

Substituting equation (35) into equation (31) results in

$$G_r(z) = \frac{\sum_{i=0}^m \bar{d}_i \left(\frac{1-z^{-1}}{1+z^{-1}} \right)^i}{\sum_{i=0}^m \bar{e}_i \left(\frac{1-z^{-1}}{1+z^{-1}} \right)^i} = \frac{\sum_{i=0}^m [\bar{d}_i (1-z^{-1})^i (1+z^{-1})^{m-i}]}{\sum_{i=0}^m [\bar{e}_i (1-z^{-1})^i (1+z^{-1})^{m-i}]} \quad (36)$$

The algorithm described in Appendix B can be used to compute the $v_{j(i,m)}$ coefficients in the following equation:

$$(1-z^{-1})^i (1+z^{-1})^{m-i} = \sum_{j=0}^m v_{j(i,m)} z^{-j} \quad (37)$$

Substituting equation (37) into equation (36) gives

$$G_r(z) = \frac{\sum_{i=0}^m \bar{d}_i \left(\sum_{j=0}^m v_{j(i,m)} z^{-j} \right)}{\sum_{i=0}^m \bar{e}_i \left(\sum_{j=0}^m v_{j(i,m)} z^{-j} \right)} = \frac{\sum_{j=0}^m \left(\sum_{i=0}^m \bar{d}_i v_{j(i,m)} \right) z^{-j}}{\sum_{j=0}^m \left(\sum_{i=0}^m \bar{e}_i v_{j(i,m)} \right) z^{-j}} = \frac{\sum_{j=0}^m d_j z^{-j}}{\sum_{j=0}^m e_j z^{-j}} \quad (38)$$

where

$$d_j = \frac{\sum_{i=0}^m \bar{d}_i v_{j(i,m)}}{\sum_{i=0}^m \bar{e}_i v_{j(i,m)}} \quad (39)$$

Recall that the e_j coefficients were determined from the dominant roots, i.e., equation (16) or the algorithm in Appendix A. Transforming the reduced z -transfer function back into the time domain gives the following reduced transfer function:

$$y_t = \sum_{j=0}^m (d_j u_{t-j\delta}) - \sum_{j=1}^m (e_j y_{t-j\delta}) \quad (40)$$

An example of dominant root model reduction for a single input transfer function is considered in Appendix C.

Reduction of Multiple-Input Transfer Functions

This section extends the single-input model reduction method described in the previous section to multiple-input transfer functions. First, a step-by-step procedure for reducing multiple-input transfer functions is presented. Then, a discussion of dominant root model reduction for building element transfer functions and CRTF's will be presented.

The following procedure can be used to compute reduced multiple-input transfer functions:

- 1 Use a root-finding procedure to determine the roots of equation (7). The roots in equation (7) are the same for all inputs because equation (7) is based upon the transfer function coefficients for outputs.

- 2 Use equation (9) to determine $n \beta_j$ terms for every input (n is equal to the number of past time steps in the original transfer function).

- 3 Use equation (13) to determine the $n \omega_j$ values for every input.

- 4 Select the dominant roots for every input. Let m equal the total number of dominant roots for all inputs.

- 5 Use equation (16) or the algorithm described in Appendix A to determine the transfer function coefficients for past outputs from the m dominant roots.

- 6 Use z -transform theory and the bilinear transformation to determine single-input w -transfer functions for every input from the original transfer function, i.e., use equations (22) and (23) and the algorithm described in Appendix B to determine the coefficients in equation (21) for every input.

- 7 Use equation (27) to determine the power series expansion of the w -transfer functions for terms of order w^0 to w^m (a power series expansion must be computed for every input).

- 8 Use equation (32) and the algorithm described in Appendix B to determine the denominator of a reduced w -transfer function. (The denominators of all the reduced transfer functions are the same.)

- 9 Use equation (34) to determine the numerator of the reduced w -transfer functions from the denominator of the reduced w -transfer function and the power series expansion of the original w -transfer functions.

- 10 Use equation (39) to determine the coefficients for the inputs in the reduced multiple-input transfer function from the reduced single-input w -transfer functions.

When using the methods of Stephenson and Mitalas (1971), Ceylan and Myers (1980), and Hittle (1981) to determine transfer functions for building elements, the roots are determined before the output transfer function coefficients are computed. Thus, a root-finding procedure is not needed when computing reduced transfer functions for building elements, i.e., step one can be eliminated. Determining the roots of a CRTF may be a numerically difficult problem. Seem (1987) presented a method for avoiding this numerical problem.

Table 1 Transfer function coefficients for ASHRAE wall 25

j	a_j (W/m ² ·°C)	b_j (W/m ² ·°C)	c_j
0	0.0021224	-4.0930276	1.0000000
1	0.0467467	6.9131200	-1.0305444
2	0.0558484	-3.0653380	0.2012205
3	0.0071115	0.1336760	-0.0072612
4	0.0000640	-0.0002212	0.0000026

Table 2 Roots and ω_j values for ASHRAE wall 25

j	λ_j	ω_j	
		Outdoor temperature	Indoor temperature
1	0.78640705	0.718	0.066
2	0.19749089	0.062	0.722
3	0.04628098	0.019	0.002
4	0.00036547	0.002	0.004

Table 3 Reduced transfer function coefficients for ASHRAE wall 25

j	a_j (W/m ² ·°C)	b_j (W/m ² ·°C)	c_j
0	0.0136599	-4.0877158	1.0000000
1	0.0128340	6.7063322	-0.9838979
2	0.0908708	-2.7359821	0.1553082

Table 4 Reduced transfer function coefficients for ASHRAE wall 25

j	Dominant root		Largest root	
	b_j (W/m ² ·°C)	c_j	b_j (W/m ² ·°C)	c_j
0	-4.30546	1.00000	-1.64838	1.00000
1	3.75600	-0.19749	1.50217	-0.78641

Applications

To test dominant root model reduction for building elements with a wide range of properties, reduced transfer functions for the following ASHRAE (1977) building elements were computed:

- 1 Exterior Wall 4 (0.1 m face brick, air space, and 0.2 m high).
- 2 Exterior Wall 25 (frame wall with 0.1 m brick veneer).
- 3 Exterior Wall 28 (metal curtain wall with 0.05 m of insulation).
- 4 Exterior Wall 36 (frame wall with 0.08 m insulation).
- 5 Exterior Wall 54 (0.1 m face brick, air space, and 0.3 m high weight concrete).

For all of the building elements tested, dominant root model reduction was used to obtain a reduced set of coefficients that closely modeled the response of the full set of coefficients.

Table 1 contains transfer function coefficients generated from Mitalas and Arsenault's (1971) program for ASHRAE wall 25. Table 2 contains the roots and values of ω_j for ASHRAE wall 25. Table 2 shows that the first root is the dominant root for a step change in the outdoor temperature and the second root is the dominant root for a step change in indoor temperature. These two dominant roots were used to obtain the reduced transfer function coefficients in Table 3. Figure 1 is a graph of the response to a 1°C step change in outdoor temperature for the full set of coefficients, the reduced set of coefficients, and the dropped set of coefficients, i.e., the full set of coefficients for two time steps back. Figure 2 is a similar graph for a 1°C step change in indoor temperature. Both these graphs demonstrate that the reduced set of coefficients closely match the response of the full set of coefficients and the dropped set of coefficients produces a response different from the full set of coefficients.

To demonstrate that the second root is dominant for the indoor temperature, reduced transfer functions were computed with both the largest root and the dominant root. (The largest

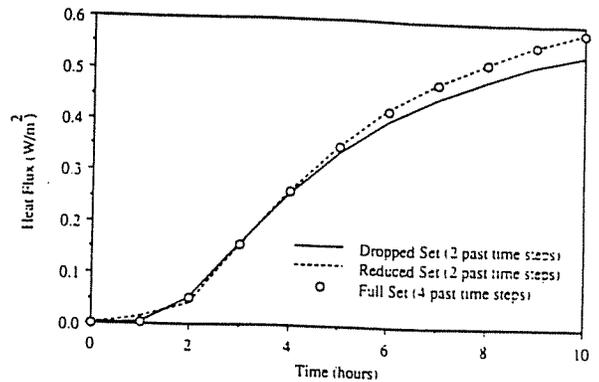


Fig. 1 Response to 1°C step change in outdoor temperature for ASHRAE wall 25

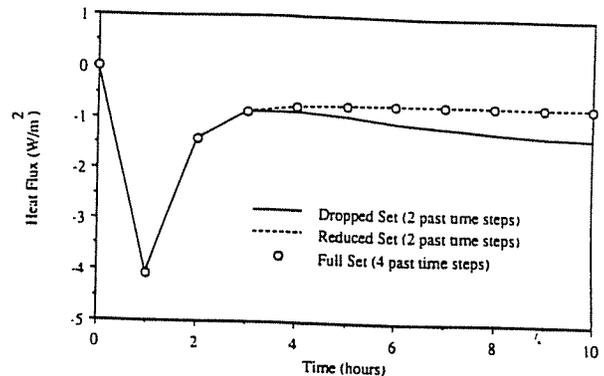


Fig. 2 Response to 1°C step change in indoor temperature for ASHRAE wall 25

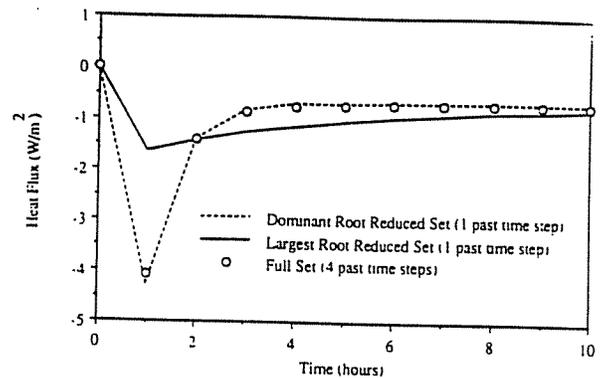


Fig. 3 Response to 1°C step change in outdoor temperature for ASHRAE wall 25

root is the root with the largest value.) Table 4 contains these reduced transfer functions. Figure 3 is a graph of the response to a 1°C step change in temperature for the full set of coefficients and reduced sets of coefficients, which were obtained with both the dominant root and the largest root. The response for the reduced transfer function with the dominant root is much closer to the response of the full set of coefficients than the response for the reduced transfer function with the largest root.

Dominant root model reduction can also be used to determine reduced CRTF's. Seem (1987) contains the original and reduced transfer function coefficients for an eight-surface room. Figure 4 shows the response to a 1°F step change in outdoor temperature for a CRTF with 19 past time steps and a reduced CRTF with 3 past time steps. The responses for the original and reduced transfer functions are nearly identical. Seem (1987) shows similar figures for step changes in indoor temperature, solar radiation gains, and radiation gains from people, equipment and lights. These figures demonstrate that dominant root model reduction can be used to reduce signif-

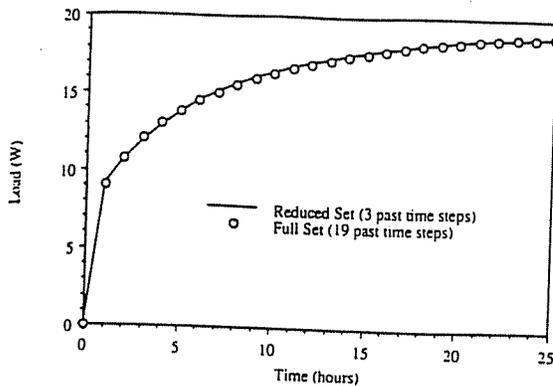


Fig. 4 Response to 1°C step change in outdoor temperature for the eight-surface room.

icantly the number of coefficients in a CRTF. Seem (1987) describes a method for determining the "correct" model order of reduced transfer functions, i.e., the minimum number of past time steps required to model the heat transfer processes accurately in a building with a reduced transfer function.

Conclusions

A new model reduction method for reducing the number of coefficients in multiple-input transfer functions with real and distinct roots has been presented in this paper. This model reduction method can be used to reduce significantly the computational effort of performing simulations of transfer functions used to solve heat transfer problems.

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APPENDIX A

Algorithm for Determining Output Coefficients

This appendix contains an algorithm that can be used to determine the output coefficients of the reduced transfer functions from m dominant roots of the original transfer function. The following equation relates the m dominant roots of the original transfer function with the output coefficients of the reduced transfer function:

$$\prod_{j=1}^m (1 - \bar{\lambda}_j z^{-1}) = \sum_{j=0}^m e_j z^{-j} \quad (41)$$

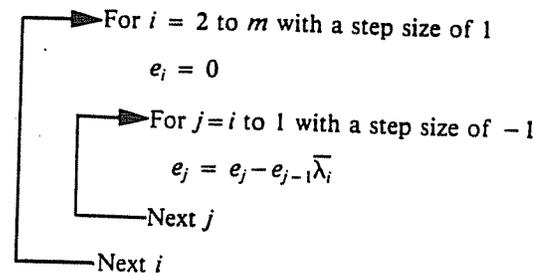
where

$$\bar{\lambda}_j = \text{one of the } m \text{ dominant roots}$$

The following algorithm can be used to determine the e_j coefficients in equation (15) or equation (41):

$$e_0 = 1$$

$$e_1 = -\bar{\lambda}_1$$



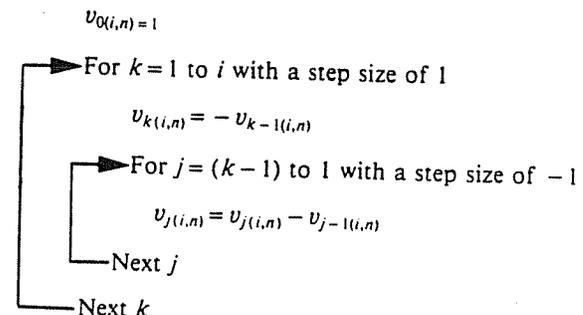
APPENDIX B

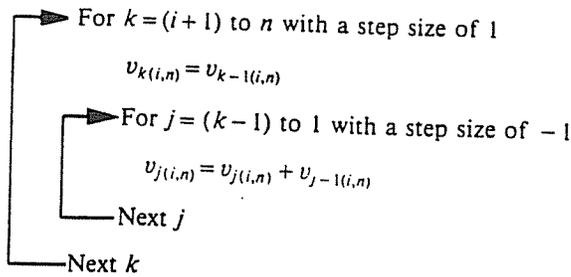
Extension of Pascal's Triangle

Numerical analysis textbooks (Conte and de Boor, 1980; Sedgewick, 1983) contain algorithms for multiplying polynomials. These algorithms could be used to obtain the $v_{j(i,n)}$ coefficients in the following equation:

$$(1-x)^i (1+x)^{n-i} = \sum_{j=0}^n v_{j(i,n)} x^j \quad (42)$$

This appendix contains a numerically efficient algorithm for determining the $v_{j(i,n)}$ coefficients in equation (42). The algorithm is numerically efficient because no multiplications or divisions are required. The following algorithm for computing the $v_{j(i,n)}$ coefficients is based upon an extension of Pascal's triangle (Spiegel, 1968):





APPENDIX C

Example

To demonstrate dominant root model reduction, a reduced transfer function with one past time step (i.e., $m=1$) will be determined from a transfer function with two past time steps (i.e., $n=2$). For this example, the following transfer function will be reduced:

$$y_t = a_0 u_t = a_1 u_{t-\delta} + a_2 u_{t-2\delta} - b_1 y_{t-\delta} - b_2 y_{t-2\delta}$$

$$= u_t + 0.5 u_{t-\delta} + 0.1 u_{t-2\delta} + 1.1 y_{t-\delta} - 0.3 y_{t-2\delta}$$

From equation (5), the z-transform of the output when the input is a unit step is

$$Y(z) = \frac{z(a_0 z^2 + a_1 z + a_2)}{(z-1)(b_0 z^2 + b_1 z + b_2)} = \frac{z(z^2 + 0.5z + 0.1)}{(z-1)(z^2 - 1.1z + 0.3)}$$

The quadratic equation can be used to determine the roots of equation (7).

$$\lambda_1, \lambda_2 = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2}$$

$$= \frac{1.1 \pm \sqrt{(-1.1)^2 - 4(0.3)}}{2} = 0.6, 0.5$$

Using the roots determined from the quadratic equation, the z-transform of the output can be written as

$$Y(z) = \frac{a_0 z^3 + a_1 z^2 + a_2 z}{(z-\lambda_0)(z-\lambda_1)(z-\lambda_2)} = \frac{z^3 + 0.5z^2 + 0.1z}{(z-1)(z-0.6)(z-0.5)}$$

From equation (9), the β_j coefficients in equation (8) are

$$\beta_1 = \frac{a_0 \lambda_1^2 + a_1 \lambda_1 + a_2}{(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)} = \frac{1(0.6)^2 + 0.5(0.6) + 0.1}{(0.6-1)(0.6-0.5)} = -19$$

$$\beta_2 = \frac{a_0 \lambda_2^2 + a_1 \lambda_2 + a_2}{(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)} = \frac{1(0.5)^2 + 0.5(0.5) + 0.1}{(0.5-1)(0.5-0.6)} = 12$$

Using equation (13), the following ω_j quantities can be computed for the roots:

$$\omega_1 = \left| \frac{\beta_1}{1-\lambda_1} \right| = \left| \frac{-19}{1-0.6} \right| = 47.5$$

$$\omega_2 = \left| \frac{\beta_2}{1-\lambda_2} \right| = \left| \frac{12}{1-0.5} \right| = 24$$

ω_1 is larger than ω_2 , therefore the dominant root is

$$\bar{\lambda}_1 = \lambda_1 = 0.6$$

Next, the coefficient in the denominator of the reduced transfer function is computed from equation (16)

$$e_1 = -\bar{\lambda}_1 = -0.6$$

From equation (18), the original w-transfer function is

$$G(w) = \frac{a_0(1+w)^2 + a_1(1-w)(1+w) + a_2(1-w)^2}{b_0(1+w)^2 + b_1(1-w)(1+w) + b_2(1-w)^2}$$

$$= \frac{(1+w)^2 + 0.5(1-w)(1+w) + 0.1(1-w)^2}{(1+w)^2 - 1.1(1-w)(1+w) + 0.3(1-w)^2}$$

$$= \frac{1.6 + 1.8w + 0.6w^2}{0.2 + 1.4w + 2.4w^2} = \frac{\bar{a}_0 + \bar{a}_1 w + \bar{a}_2 w^2}{\bar{b}_0 + \bar{b}_1 w + \bar{b}_2 w^2}$$

The first two terms in the power series expansion of $G(w)$ can be computed from equation (27)

$$\bar{c}_0 = \frac{\bar{a}_0}{\bar{b}_0} = \frac{1.6}{0.2} = 8$$

$$\bar{c}_1 = \frac{\bar{a}_1 - \bar{b}_1 \bar{c}_0}{\bar{b}_0} = \frac{1.8 - (1.4)(8)}{0.2} = -47$$

From equation (28), the reduced w-transfer function is

$$G_r(w) = \frac{N_r(w)(1+w)}{e_0(1+w) + e_1(1-w)}$$

$$= \frac{N_r(w)(1+w)}{(1+w) + (-0.6)(1-w)}$$

$$= \frac{N_r(w)(1+w)}{0.4 + 1.6w} = \frac{N_r(w)(1+w)}{\bar{e}_0 + \bar{e}_1 w}$$

The numerator coefficients of the reduced w-transfer function are determined from equation (34)

$$\bar{d}_0 = \bar{e}_0 \bar{c}_0 = (0.4)(8) = 3.2$$

$$\bar{d}_1 = \bar{e}_0 \bar{c}_1 + \bar{e}_1 \bar{c}_0 = (0.4)(-47) + (1.6)(8) = -6$$

Equation (36) can be used to compute the reduced z-transfer function from the reduced w-transfer function

$$G_r(z) = \frac{\bar{d}_0(1+z^{-1}) + \bar{d}_1(1-z^{-1})}{\bar{e}_0(1+z^{-1}) + \bar{e}_1(1-z^{-1})}$$

$$= \frac{3.2(1+z^{-1}) - 6(1-z^{-1})}{0.4(1+z^{-1}) + 1.6(1-z^{-1})}$$

$$= \frac{-1.4 + 4.6z^{-1}}{1 - 0.6z^{-1}}$$

Transforming the reduced z-transfer function into the time domain gives

$$y_t = -1.4 u_t + 4.6 u_{t-\delta} + 0.6 y_{t-\delta}$$