

Optimum Heat Power Cycles for Specified Boundary Conditions

O. M. Ibrahim

S. A. Klein

J. W. Mitchell

Solar Energy Laboratory,
University of Wisconsin-Madison,
Madison, WI 53706

Optimization of the power output of Carnot and closed Brayton cycles is considered for both finite and infinite thermal capacitance rates of the external fluid streams. The method of Lagrange multipliers is used to solve for working fluid temperatures that yield maximum power. Analytical expressions for the maximum power and the cycle efficiency at maximum power are obtained. A comparison of the maximum power from the two cycles for the same boundary conditions, i.e., the same heat source/sink inlet temperatures, thermal capacitance rates, and heat exchanger conductances, shows that the Brayton cycle can produce more power than the Carnot cycle. This comparison illustrates that cycles exist that can produce more power than the Carnot cycle. The optimum heat power cycle, which will provide the upper limit of power obtained from any thermodynamic cycle for specified boundary conditions and heat exchanger conductances is considered. The optimum heat power cycle is identified by optimizing the sum of the power output from a sequence of Carnot cycles. The shape of the optimum heat power cycle, the power output, and corresponding efficiency are presented. The efficiency at maximum power of all cycles investigated in this study is found to be equal to (or well approximated by) $\eta = 1 - \sqrt{T_{L,in}/\phi T_{H,in}}$ where ϕ is a factor relating the entropy changes during heat rejection and heat addition.

Introduction

Carnot (1824) introduced the concepts of reversibility and the principle that the thermal efficiency of a reversible cycle is determined solely by the temperatures of the heat source and heat sink. The maximum possible efficiency is obtained when only reversible processes are involved. Reversible heat transfer processes require thermal energy to be transferred with infinitesimal temperature differences. For finite heat exchange areas and heat transfer coefficients, the rate of heat exchange in reversible processes approaches zero, and the total energy transfer in finite time is zero. As a result, as a cycle approaches thermodynamic reversibility, its output power becomes infinitesimally small.

Consider a simplified model of a Carnot cycle¹ where all of the irreversibilities are associated with the transfer of heat to and from the power cycle equipment. There are no internal losses within the cycle itself. The heat source and heat sink are first considered to be at constant temperatures T_H and T_L , respectively. Because of the finite heat transfer conductance of materials, the cycle operates between T_h and T_l as shown in Fig. 1 to provide temperature differences, the driving forces for heat transfer between the cycle and the hot and cold reservoirs. Figure 2 shows a plot of power versus efficiency for

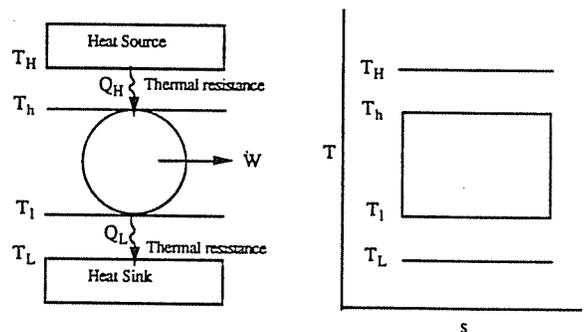


Fig. 1 Carnot cycle coupled to heat source and heat sink with infinite heat capacitance rates

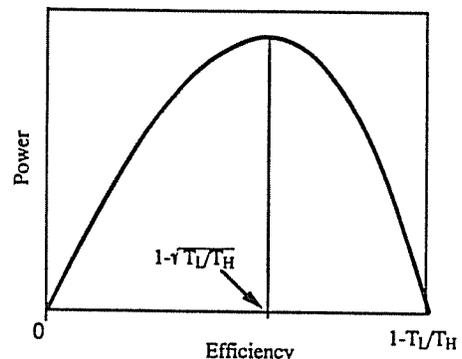


Fig. 2 Power/efficiency tradeoffs for the Carnot cycle

¹In this paper, the term "Carnot cycle" is used to refer to a cycle having two adiabatic and two isothermal processes. The processes may be irreversible and as a result the cycle may not achieve the Carnot cycle efficiency.

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the cycle. Finite power can only be achieved by operating at less than the maximum efficiency. A point of maximum power exists, and at this point the efficiency is given by

$$\eta = 1 - \sqrt{T_L/T_H} \quad (1)$$

The existence of a maximum power point for this simple cycle and the limitation of the efficiencies of real processes resulting from finite heat transfer rate constraints were recognized by El-Wakil (1962), and by Curzon and Ahlborn (1975). They considered a Carnot cycle with heat transfer limitations and derived expressions for the maximum power and the cycle efficiency at the maximum power. Their studies were limited to the case in which the heat transfer fluid streams have infinite thermal capacitance rates (mass flow rate – specific heat product), i.e., the source and sink were assumed to be isothermal. Curzon and Ahlborn's study has inspired many subsequent studies. Leff (1987a) shows that the Brayton, Otto, Diesel and Atkinson cycles also produce maximum power at an efficiency of $\eta = 1 - \sqrt{T_L/T_H}$. Gordon (1988) has investigated the maximum power-efficiency relations for solar-driven cycles. Bejan (1988) considered an irreversible power plant model, accounting for the heat loss through the plant to the surroundings. Several papers deal with optimum power of the Carnot cycle operating between finite thermal capacitance rates heat source and sink, e.g., Ondrechen et al. (1983) and Wu (1988). Lee et al. (1990) analytically present the optimum power and the efficiency of a finite time Carnot heat engine operating between two reservoirs with finite heat-capacity rates. They show that the efficiency at maximum power depends only on the inlet temperatures of the cold and hot reservoirs. To achieve their results, a periodic cycle is considered with heat exchange contact time being among the independent variables. Rubin (1979a) extended the Curzon and Ahlborn model by consideration of cycles in which the reservoir temperatures are controllable parameters varying between the temperatures of the coldest and hottest reservoirs. Using a Lagrange optimization method, he shows that the efficiency at maximum power depends only on the coldest and hottest reservoir temperatures. Rubin (1979b) considered a reciprocating engine with heat exchange contact time being among the independent variables. The engine is a standard engine with a cylinder and piston, which is used to do work on the outside world. The working fluid is a perfect gas. He used optimum-control theory to find the possible optimum controls and the optimum trajectories for this type of engine.

Ondrechen et al. (1981) investigated a reversible cycle using an ideal gas working fluid providing maximum power using a finite thermal capacitance rate heat source. They considered

a series of sequential Carnot cycles where each sequential cycle operates at its maximum power point. The heat sink was assumed to be isothermal and the heat exchangers were assumed infinitely large. Leff (1987b) used a second law availability analysis to obtain the maximum work available for the case where both heat source and heat sink have finite thermal capacitance rates. However, no study has identified the nature of a power cycle that produces maximum power for the case where heat exchanger area and heat transfer coefficients are finite, the cycle is internally irreversible, and the working fluid is not restricted to an ideal gas.

Although the Carnot cycle is the most efficient cycle, there exist cycles that can produce more power than the Carnot cycle for the same boundary conditions when heat transfer constraints are considered. The goal of this paper is to identify the optimum power cycle, i.e., the cycle that produces the maximum power for specified external boundary conditions. Although still somewhat idealized, the cycle that provides maximum power for specified external boundary conditions gives a more reasonable design goal than do reversible cycles that generate zero power. In this analysis, heat exchanger areas and heat transfer coefficients are finite and the cycle may be internally irreversible. Analytical and numerical techniques are used to determine the maximum power. The shape of optimum power cycle, i.e., the working fluid temperature as a function of entropy, $T(s)$, along with power output and efficiency, are presented for given heat source, heat sink, and heat exchanger conductances.

Irreversible Power Cycle Model

The irreversibilities that constrain the performance of a power cycle can be considered as external and internal. External irreversibilities arise from temperature differences that occur between the cycle and the heat source and sink. Internal irreversibilities result primarily because of fluid and mechanical friction. External irreversibilities are considered in previous studies (e.g., Curzon and Ahlborn, 1975).

There are several ways to account for internal irreversibilities. Howe (1982) derived an equation for the efficiency at maximum power for an irreversible Carnot cycle, accounting for internal irreversibilities, by multiplying the thermal efficiency of a reversible cycle by a factor (ψ) less than one (equal to one when the cycle is internally reversible).

$$\eta = \psi[1 - T_l/T_h] \quad (2)$$

El-Wakil (1962) accounted for internal irreversibilities by relating the entropy change during heat rejection for reversible and irreversible cycles.

Nomenclature

C_p = specific heat capacity at constant pressure, kJ/kg K	s = entropy, kJ/kg K	Subscripts
C_v = specific heat capacity at constant volume, kJ/kg K	\dot{S} = entropy transfer rate (mass flow rate-entropy product), kW/K	A = area
\dot{C} = heat capacitance rate (mass flow rate-specific heat product), kW/K	T = temperature, K	B = Brayton
k = heat capacity ratio = C_p/C_v	UA = heat exchanger conductance (total heat transfer coefficient area product), kW/K	C = Carnot
i = fraction of unavailable energy that occurs with internal irreversibilities (El-Wakil, 1962)	\dot{W} = power, kW	H = heating fluid, heat source, hot-side heat exchanger
\dot{m} = mass flow rate, kg/s	ϵ = heat exchanger effectiveness	h = high
N = number of cycles	η = thermal efficiency	L = cooling fluid, heat sink, cold-side heat exchanger
NTU = number of transfer units = UA/C	λ = Lagrange multiplier	l = low
\dot{Q} = rate of heat transfer, kW	ψ = a factor relating the efficiency of an irreversible cycle to a reversible one (Howe, 1982)	i = cycle
P = pressure, N/m ²	ϕ = a factor relating the entropy change during heat rejection and heat addition	in = inlet
		min = minimum
		max = maximum
		op = optimum
		p = production
		$total$ = total
		wf = working fluid

$$\Delta \dot{S}_L / \Delta \dot{S}_L = 1 + i \quad (3)$$

where i = fraction of unavailability energy that occurs with internal irreversibilities; $\Delta \dot{S}_L$ = entropy change during heat rejection for irreversible cycles; ΔS_L = entropy change during heat rejection for reversible cycles. Equation (3) leads to the following expression for the thermal efficiency for an irreversible cycle:

$$\eta = 1 - (i+1)T_l/T_h \quad (4)$$

Thermodynamic analysis of a Carnot cycle results in

$$\dot{W} = \dot{Q}_H - \dot{Q}_L \quad (5)$$

$$\frac{\dot{Q}_H}{T_h} - \frac{\dot{Q}_L}{T_l} \leq 0 \quad (6)$$

where \dot{Q}_L is the rate of heat rejected, \dot{Q}_H is the rate of heat supplied, and \dot{W} is the power output. Equation (6) is known as the "Clausius Inequality."

In this study, a simple model relating the entropy change during heat rejection and heat addition is considered. An irreversibility factor (ϕ) is defined such that the entropy inequality for internally irreversible cycles, Eq. (6), can be written as an equality:

$$\frac{\dot{Q}_H}{T_h} - \phi \frac{\dot{Q}_L}{T_l} = 0 \quad (7)$$

ϕ is equal to one when the cycle is internally reversible and less than one when the cycle is internally irreversible. The entropy production, \dot{S}_p , is related to ϕ by the following equation:

$$\dot{S}_p = (1 - \phi) \frac{\dot{Q}_L}{T_l} \quad (8)$$

Introducing ϕ makes it possible to derive an analytical solution for an internally irreversible Carnot cycle similar to the solution obtained for the internally reversible one. In the following analysis, ϕ is assumed to be constant. The thermal efficiency for the irreversible cycle then can be written as

$$\eta = 1 - \frac{T_l}{\phi T_h} \quad (9)$$

The fact that η should be greater than zero requires that $\phi \geq T_l/T_h$, which sets a lower limit for ϕ . All of the above expressions for the thermal efficiency for irreversible cycles are related in the following manner:

$$\phi = \frac{1}{1+i} \quad (10)$$

$$\psi = \frac{1 - T_l/\phi T_h}{1 - T_l/T_h} \quad (11)$$

Maximum Power of a Carnot Cycle

Carnot Cycle Coupled to Heat Source and Sink With Infinite Thermal Capacitance Rates. When the Carnot cycle is coupled to an isothermal heat source and sink as shown in Fig. 3, the rates at which heat is supplied and rejected are given by:

$$\dot{Q}_H = UA_H(T_H - T_h) \quad (12)$$

$$\dot{Q}_L = UA_L(T_l - T_L) \quad (13)$$

where UA_H and UA_L are, respectively, the hot-side and cold-side heat exchanger conductance (heat transfer coefficient-area product). The energy and the entropy balances on the cycle are then given by

$$\dot{W} = UA_H(T_H - T_h) - UA_L(T_l - T_L) \quad (14)$$

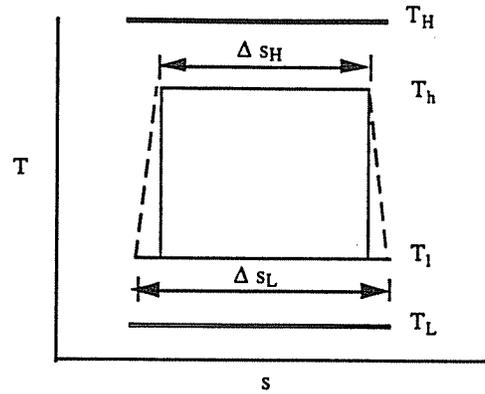


Fig. 3 Carnot cycle with irreversible expansion and compression coupled to heat source and heat sink with infinite heat capacitance rates

$$g = \frac{UA_H(T_H - T_h)}{T_h} - \phi \frac{UA_L(T_l - T_L)}{T_l} = 0 \quad (15)$$

where g is the entropy constraint function.

The values of T_h and T_l that result in the maximum power are found using the method of Lagrange multipliers. A Lagrange multiplier, λ , is defined such that:

$$\partial \dot{W} / \partial T_h = \lambda \partial g / \partial T_h \quad (16)$$

$$\partial \dot{W} / \partial T_l = \lambda \partial g / \partial T_l \quad (17)$$

Evaluating the partial derivatives allows Eqs. (16) and (17) to be written as:

$$1 = \lambda T_H / T_h^2 \quad (18)$$

$$1 = \lambda \phi T_L / T_l^2 \quad (19)$$

Solving Eqs. (15), (18), and (19) leads to the following relation for the unknown cycle temperatures:

$$T_l / T_h = \sqrt{\phi T_L / T_H} \quad (20)$$

The efficiency at the maximum power point is then given by:

$$\eta = 1 - \sqrt{T_L / (\phi T_H)} \quad (21)$$

The values of T_h and T_l that result in the maximum power are:

$$T_h = \left[\frac{\phi UA_L \sqrt{T_L / \phi} + UA_H \sqrt{T_H}}{\phi UA_L + UA_H} \right] \sqrt{T_H} \quad (22)$$

$$T_l = \left[\frac{\phi UA_L \sqrt{T_L / \phi} + UA_H \sqrt{T_H}}{\phi UA_L + UA_H} \right] \sqrt{\phi T_L} \quad (23)$$

The maximum power for a single Carnot cycle coupled to an isothermal heat source and heat sink is then:

$$\dot{W}_{\max} = \frac{\phi UA_L}{\phi UA_L + UA_H} UA_H [\sqrt{T_H} - \sqrt{T_L / \phi}]^2 \quad (24)$$

The Lagrange multiplier method locates an extremum point, which may be a minimum or a maximum. A verification that \dot{W}_{\max} in Eq. (24) is a maximum is possible by obtaining the sign of the second derivative of \dot{W}_{\max} with respect to any free variable (e.g., T_h , T_l) or by examination of a plot of \dot{W}_{\max} versus the variable, as in Fig. 4.

Equation (21) shows that the efficiency of a Carnot cycle at a maximum power depends only on the temperature of the heat reservoirs and the irreversibility factor ϕ . If $\phi = 1$, the efficiency reduces to the relationship of El-Wakil (1962), and Curzon and Ahlborn (1975). The effects of internal irreversibility on the power/efficiency tradeoffs are shown in Fig. 4. The internal irreversibilities decrease both the maximum power and the efficiency at the maximum power.

The variation of power with the cold-side heat exchanger conductance ratio, defined as the fraction of total heat ex-

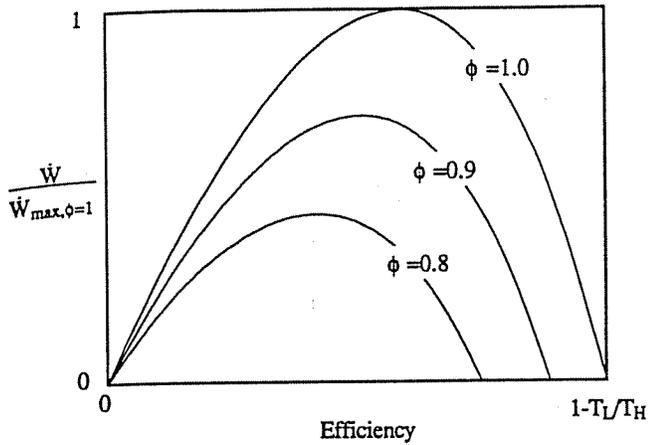


Fig. 4 Power/efficiency tradeoffs for the Carnot cycle showing the effects of internal irreversibility ($UA_H = UA_L$)

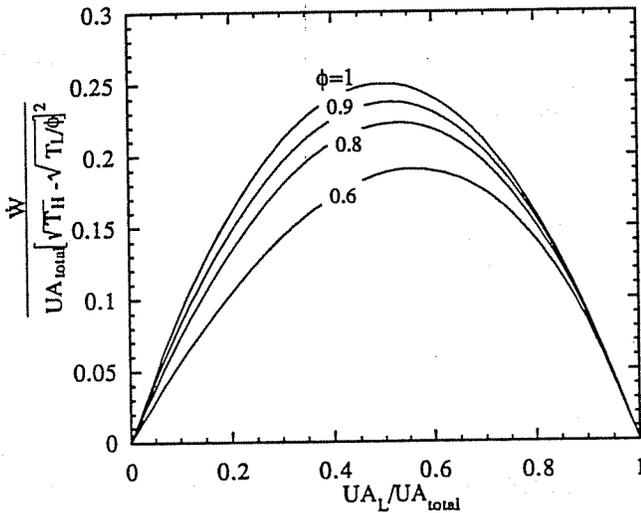


Fig. 5 Variation of power output with cold-side heat exchanger conductance ratio

changer conductance used at the cold-side heat exchanger, is shown in Fig. 5. There exists an optimum balance between the conductances of the hot-side and cold-side heat exchangers. The total heat exchanger conductance, $UA_{total} = UA_L + UA_H$, should be split evenly when the cycle is internally reversible, as concluded by Bejan (1988). However, as the internal irreversibility increases, the cold-side heat exchanger should be increasingly larger than the hot-side heat exchanger to obtain maximum power because the internal irreversibility increases the heat rejection. A larger cold-side heat exchanger reduces the external irreversibility by allowing the cycle to reject energy at a lower temperature.

Carnot Cycle Coupled to Heat Source and Sink With Finite Thermal Capacitance Rates. When the Carnot cycle is coupled to a source and sink with finite thermal capacitance rates, the rates at which heat is supplied and rejected can be expressed as (Kays and London, 1964)

$$\dot{Q}_H = \dot{C}_H \epsilon_H (T_{H,in} - T_h) \quad (25)$$

$$\dot{Q}_L = \dot{C}_L \epsilon_L (T_l - T_{L,in}) \quad (26)$$

where $T_{H,in}$ = inlet temperature of the heating fluid; $T_{L,in}$ = inlet temperature of the cooling fluid; \dot{C}_H = thermal capacitance rate of the heating fluid; \dot{C}_L = heat capacitance rate of the cooling fluid; ϵ_H = effectiveness of the hot-side heat exchanger = $1 - \exp(-NTU_H)$; ϵ_L = effectiveness of the cold-side heat ex-

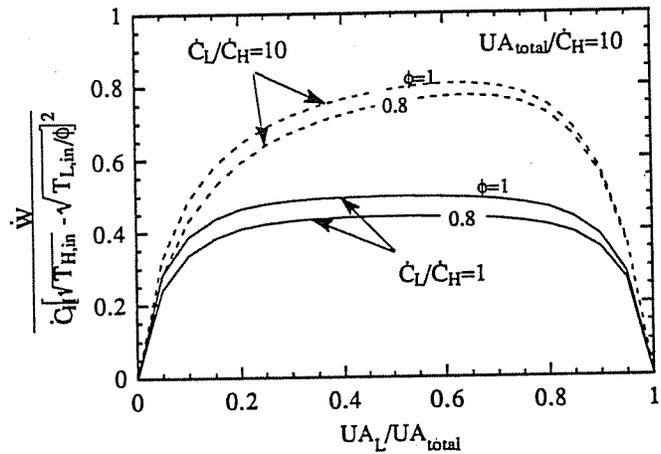


Fig. 6 Variation of power output with cold-side heat exchanger conductance ratio

changer = $1 - \exp(-NTU_L)$. Using the method of Lagrange multipliers, similar to the analysis for infinite thermal capacitance rates, the optimum power for a single Carnot cycle coupled to source and sink streams with finite thermal capacitance rates is

$$\dot{W}_{max} = \frac{\phi \dot{C}_L \epsilon_L}{\phi \dot{C}_L \epsilon_L + \dot{C}_H \epsilon_H} \dot{C}_H \epsilon_H [\sqrt{T_{H,in}} - \sqrt{T_{L,in}/\phi}]^2 \quad (27)$$

Equation (27) reduces to Eq. (24) in the limit of high capacity rates whereupon $\epsilon_H \dot{C}_H = UA_H$ and $\epsilon_L \dot{C}_L = UA_L$.

The efficiency at the maximum power point is given by:

$$\eta = 1 - \sqrt{T_{L,in}/\phi T_{H,in}} \quad (28)$$

Equations (21) and (28), which are very similar, demonstrate that for both finite and infinite capacitance rates, the efficiency of a Carnot cycle at maximum power depends only on the inlet temperatures of the heat reservoirs.

The variation of the power with the cold-side heat exchanger conductance ratio is shown in Fig. 6. The optimum ratio is a function of the irreversibility factor, the thermal capacitance rates of the heat source and heat sink, and the total conductance. As the internal irreversibilities increase, the optimum cold-side heat exchanger conductance ratio increases. When $\dot{C}_L/\dot{C}_H = 1$, the total heat exchanger conductance, UA_{total} , should be split evenly when the cycle is internally reversible; however, as \dot{C}_L/\dot{C}_H increases, the optimum cold-side conductance ratio (UA_L/UA_{total}) increases.

Maximum Power of a Closed Brayton Cycle

A simple closed Brayton cycle, composed of two adiabatic and two constant-pressure processes, is considered with an ideal gas having constant thermal capacitance rate (\dot{C}_{wf}). The shape of the closed Brayton cycle on temperature-entropy coordinates is a function of the thermal capacitance rate of the working fluid, \dot{C}_{wf} . As \dot{C}_{wf} becomes infinitely large, the Brayton cycle maximum power approaches the Carnot cycle maximum power, since the constant pressure processes approach isothermal behavior. The effect of the Brayton cycle shape on cycle power output is presented in this section to show that there exist cycles that can produce more power than the Carnot cycle for the same boundary conditions.

Brayton Cycle Coupled to Heat Source and Sink With Infinite Thermal Capacitance Rates. The Brayton cycle coupled to heat source and heat sink with infinite thermal capacitance rates is sketched in Fig. 7. The rates at which heat is supplied and rejected are given by:

$$\dot{Q}_H = \dot{C}_{wf} \epsilon_H (T_H - T_1) = \dot{C}_{wf} (T_2 - T_1) \quad (29)$$

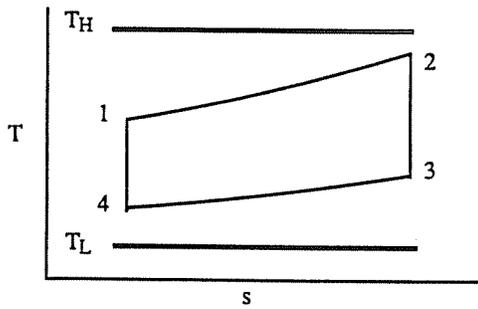


Fig. 7 Brayton cycle coupled to heat source and sink with infinite heat capacitance rates

$$\dot{Q}_L = \dot{C}_{wf} \epsilon_L (T_3 - T_L) = \dot{C}_{wf} (T_3 - T_4) \quad (30)$$

Thermodynamic analysis of a closed Brayton cycle provides

$$\dot{W} = \dot{C}_{wf} \epsilon_H (T_H - T_1) - \dot{C}_{wf} \epsilon_L (T_3 - T_L) \quad (31)$$

$$\dot{C}_{wf} \ln[T_2/T_1] - \dot{C}_{wf} \ln[T_3/T_4] = 0 \quad (32)$$

Using the method of Lagrange multipliers, the maximum power and the efficiency at maximum power for a closed internally reversible Brayton cycle coupled to an isothermal heat source and heat sink are given by:

$$\dot{W}_{\max} = \frac{\dot{C}_{wf} \epsilon_H \epsilon_L}{\epsilon_H + \epsilon_L - \epsilon_H \epsilon_L} [\sqrt{T_H} - \sqrt{T_L}]^2 \quad (33)$$

$$\eta = 1 - \sqrt{T_L/T_H} \quad (34)$$

The efficiency of the Brayton cycle can be expressed in terms of pressure ratio (P_L/P_H) only:

$$\eta = 1 - [P_L/P_H]^{(k-1)/k} \quad (35)$$

Therefore, at the maximum power point:

$$\sqrt{T_L/T_H} = [P_L/P_H]^{(k-1)/k} \quad (36)$$

A comparison of the maximum power of this Brayton cycle and the Carnot cycle is shown in Fig. 8. As C_{wf} increases, the maximum power of the Brayton cycle increases and asymptotically approaches the Carnot cycle maximum power. The maximum power of the Carnot cycle is always greater than the maximum power of Brayton cycle for the case where the heat source and heat sink are isothermal. The Carnot cycle provides better temperature matching with the isothermal heat source and heat sink, which reduces the external irreversibility.

Brayton Cycle Coupled to Heat Source and Sink With Finite Thermal Capacitance Rates. When the Brayton cycle is coupled to a heat source and sink with finite thermal capacitance rates (Fig. 9), the rates at which heat is supplied and rejected are given by:

$$\dot{Q}_H = \dot{C}_{H,\min} \epsilon_H (T_{H,\text{in}} - T_1) = \dot{C}_{wf} (T_2 - T_1) \quad (37)$$

$$\dot{Q}_L = \dot{C}_{L,\min} \epsilon_L (T_3 - T_{L,\text{in}}) = \dot{C}_{wf} (T_3 - T_4) \quad (38)$$

where the effectivenesses of hot-side heat exchanger ϵ_H , and cold-side heat exchanger ϵ_L , for counterflow heat exchangers are defined as (Kays and London, 1964):

$$\epsilon_H = \frac{1 - \exp[-NTU_H(1 - \dot{C}_{H,\min}/\dot{C}_{H,\max})]}{1 - (\dot{C}_{H,\min}/\dot{C}_{H,\max}) \exp[-NTU_H(1 - \dot{C}_{H,\min}/\dot{C}_{H,\max})]} \quad (39)$$

$$\epsilon_L = \frac{1 - \exp[-NTU_L(1 - \dot{C}_{L,\min}/\dot{C}_{L,\max})]}{1 - (\dot{C}_{L,\min}/\dot{C}_{L,\max}) \exp[-NTU_L(1 - \dot{C}_{L,\min}/\dot{C}_{L,\max})]} \quad (40)$$

where $\dot{C}_{H,\min}$ and $\dot{C}_{H,\max}$ are, respectively, the smaller and the larger of the two capacitance rates \dot{C}_H and \dot{C}_{wf} . $\dot{C}_{L,\min}$ and $\dot{C}_{L,\max}$ are, respectively, the smaller and the larger of \dot{C}_L and

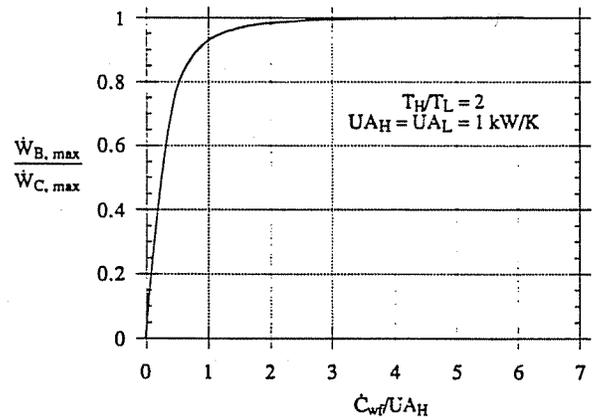


Fig. 8 Comparison of the maximum power of internally reversible Brayton and Carnot cycles when the heat source and the heat sink heat capacitance rates are infinite

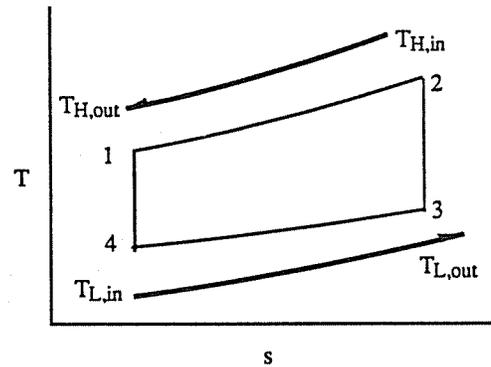


Fig. 9 Brayton cycle coupled to heat source and sink with finite heat capacitance rates

\dot{C}_{wf} . The number of heat transfer units, NTU_H and NTU_L , are based on the minimum thermal capacitance rates.

Energy and entropy balances on an internally reversible closed Brayton cycle are:

$$\dot{W} = \dot{C}_{H,\min} \epsilon_H (T_{H,\text{in}} - T_1) - \dot{C}_{L,\min} \epsilon_L (T_3 - T_{L,\text{in}}) \quad (41)$$

$$\dot{C}_{wf} \ln[T_2/T_1] - \dot{C}_{wf} \ln[T_3/T_4] = 0 \quad (42)$$

Using the method of Lagrange multipliers, similar to the analysis for the case of isothermal heat reservoirs, the optimum power and the efficiency at maximum power:

$$\dot{W}_{\max} = \frac{\dot{C}_{H,\min} \epsilon_H \dot{C}_{L,\min} \epsilon_L [\sqrt{T_{H,\text{in}}} - \sqrt{T_{L,\text{in}}}]^2}{\dot{C}_{H,\min} \epsilon_H + \dot{C}_{L,\min} \epsilon_L - \dot{C}_{H,\min} \epsilon_H \dot{C}_{L,\min} \epsilon_L / \dot{C}_{wf}} \quad (43)$$

$$\eta = 1 - \sqrt{T_{L,\text{in}}/T_{H,\text{in}}} \quad (44)$$

The effect of the heat capacitance rate of the working fluid (\dot{C}_{wf}) on the maximum power of the internally reversible Brayton cycle is shown in Fig. 10. The Brayton cycle can produce more power than the maximum power of the Carnot cycle for the same boundary conditions and the same heat exchanger conductances. The increased power of the Brayton cycle can be attributed to the lower entropy production during the heat transfer processes, due to more favorable matching of the working fluid and external stream temperatures. The optimum value of working fluid thermal capacitance rate, $\dot{C}_{wf,op}$, is always bounded by $\dot{C}_L \geq \dot{C}_{wf,op} \geq \dot{C}_H$. When $\dot{C}_L = \dot{C}_H$, $\dot{C}_{wf,op}/\dot{C}_H = 1$, the maximum power of the Brayton cycle asymptotically approaches the Carnot maximum power as \dot{C}_{wf} increases.

Optimum Heat Power Cycles

It has been shown that the Brayton cycle can produce more

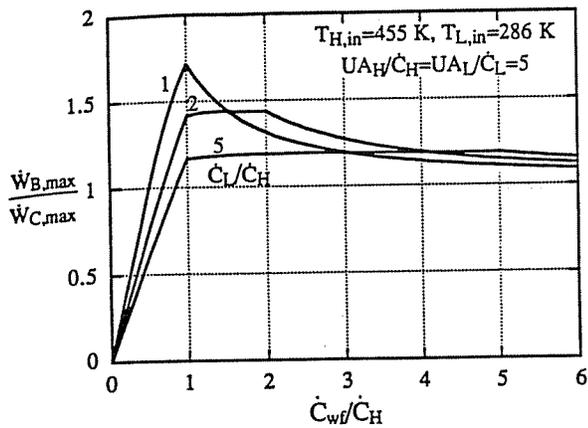


Fig. 10 Comparison of the maximum power of internally reversible Brayton and Carnot cycles when the heat source and the heat sink heat capacitance rates are finite

power than an equivalent Carnot cycle when the heat source and heat sink have finite thermal capacitance rates. Other cycles, such as the Otto and Diesel cycles, can also produce more power than the Carnot cycle. The purpose of this section is to identify the internally reversible cycle that will result in the upper limit of the maximum power for specified boundary conditions.

Any internally reversible thermodynamic cycle can be broken into a sequence of internally reversible Carnot cycles as shown in Fig. 11. An infinite number of infinitesimal cycles have the same heat interactions with the heat source and heat sink, and the same power output as the original cycle. Starting from a sequence of Carnot cycles, the optimum power cycle can be identified by dividing the total heat exchange conductance equally between the cycles. As the number of cycles in sequence approaches infinity, the performance and shape of the sequence approaches the performance and shape of the optimum power cycle. Ondrechen et al. (1981) considered sequential Carnot cycles to find the maximum power from a heat source with finite thermal capacitance rates and infinitely large heat exchangers. However, they determined the sum of the maximum power for each cycle in the sequence. Operating each cycle in the sequence at maximum power does not necessarily result in the system consisting of all cycles in the sequence producing maximum power. In this study, the goal is to optimize the sum of the power output from all cycles, rather than optimizing the power output from each individual cycle in the sequence as done by Ondrechen et al.

An energy balance on all the Carnot cycles in Fig. 11 is

$$\dot{W} = \sum_i^N [\dot{Q}_{H,i} - \dot{Q}_{L,i}] \quad (45)$$

An entropy balance for each cycle i in the sequence can be expressed as

$$\frac{\dot{Q}_{H,i}}{T_{h,i}} - \frac{\dot{Q}_{L,i}}{T_{l,i}} = 0 \quad [i = 1, N] \quad (46)$$

Optimum Power Cycle Coupled to Heat Source and Sink With Infinite Thermal Capacitance Rates. When sequential Carnot cycles are coupled to an isothermal heat source and heat sink, the power from the N -Carnot cycles is given by:

$$\dot{W} = \sum_{i=1}^N [UA_{H,i}(T_H - T_{h,i}) - UA_{L,i}(T_{l,i} - T_L)] \quad (47)$$

The shape of the optimum cycle is determined by maximizing \dot{W} with respect to $T_{h,i}$ and $T_{l,i}$, subject to the following constraints:

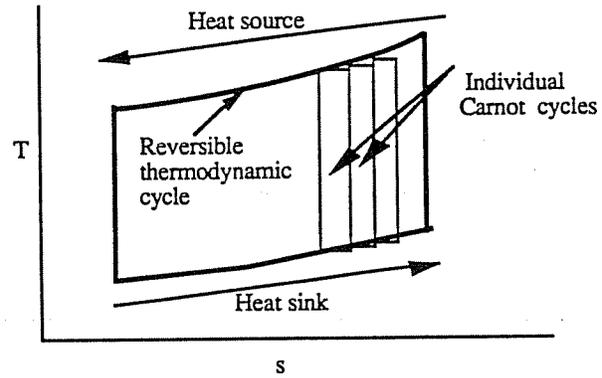


Fig. 11 A thermodynamic cycle can be broken into a sequence of Carnot cycles

$$\frac{UA_{H,i}(T_H - T_{h,i})}{T_{h,i}} - \frac{UA_{L,i}(T_{l,i} - T_L)}{T_{l,i}} = 0 \quad [i = 1, N] \quad (48)$$

Using the method of Lagrange multipliers, the efficiency at the maximum power point for any cycle i in the sequence is found, as discussed previously, to be

$$\eta = 1 - \sqrt{T_L/T_H} \quad (49)$$

The values of $T_{h,i}$ and $T_{l,i}$ that result in the maximum power are:

$$T_{h,1} = T_{h,2} = \dots = T_{h,i} = \left[\frac{UA_L \sqrt{T_L} + UA_H \sqrt{T_H}}{UA_L + UA_H} \right] \sqrt{T_H} \quad (50)$$

$$T_{l,1} = T_{l,2} = \dots = T_{l,i} = \left[\frac{UA_L \sqrt{T_L} + UA_H \sqrt{T_H}}{UA_L + UA_H} \right] \sqrt{T_L} \quad (51)$$

The maximum power for sequential Carnot cycles coupled to an isothermal heat source and heat sink is then:

$$\begin{aligned} \dot{W}_{\max} &= \sum_i^N \frac{UA_{L,i}}{UA_{L,i} + UA_{H,i}} UA_{H,i} [\sqrt{T_H} - \sqrt{T_L}]^2 \\ &= \frac{UA_L}{UA_L + UA_H} UA_H [\sqrt{T_H} - \sqrt{T_L}]^2 \quad (52) \end{aligned}$$

Equations (22), (23), (50), and (51) show that all individual cycles have the same upper and lower temperatures, and they are the same as the upper and lower temperatures resulting from the power optimization of a single cycle. The maximum power, efficiency, and the shape of N -cycles in a sequence are the same as those of a single cycle. For an isothermal heat source and heat sink, the maximum power cycle is the Carnot cycle. This conclusion is consistent with the results shown in Fig. 8, which show that the Brayton cycle always produces less power than the maximum power of the Carnot cycle for the case where the heat source and heat sink are isothermal.

Optimum Power Cycle Coupled to Heat Source and Heat Sink With Finite Thermal Capacitance Rates. When the sequential Carnot cycles are coupled to a heat source and sink with finite thermal capacitance rates, the power from the N -Carnot cycles is given by:

$$\dot{W} = \sum_{i=1}^N [\dot{C}_H \epsilon_H (T_{H,in,i} - T_{h,i}) - \dot{C}_L \epsilon_L (T_{l,i} - T_{L,in,i})] \quad (53)$$

The shape of the optimum cycle is determined by maximizing \dot{W} with respect to $T_{h,i}$ and $T_{l,i}$ subject to the following constraints:

$$\frac{\dot{C}_H \epsilon_H (T_{H,in,i} - T_{h,i})}{T_{h,i}} - \frac{\dot{C}_L \epsilon_L (T_{l,i} - T_{L,in,i})}{T_{l,i}} = 0 \quad (i = 1, N) \quad (54)$$

$$T_{H,in,i+1} = T_{H,in,i} - \epsilon_H (T_{H,in,i} - T_{h,i}) \quad (i = 1, N-1) \quad (55)$$

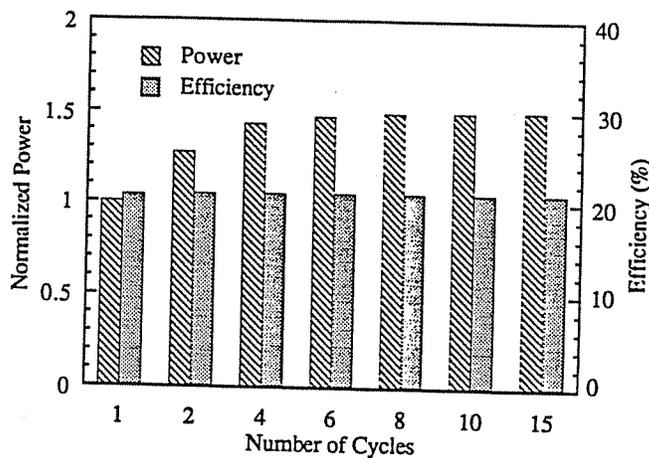


Fig. 12 Maximum power and corresponding efficiency versus the number of Carnot cycles in sequence for $T_{L,in}=286$ K, $T_{H,in}=455$, $C_L/C_H=10$, and $UA_H=UA_L$ ($NTU_H=10$)

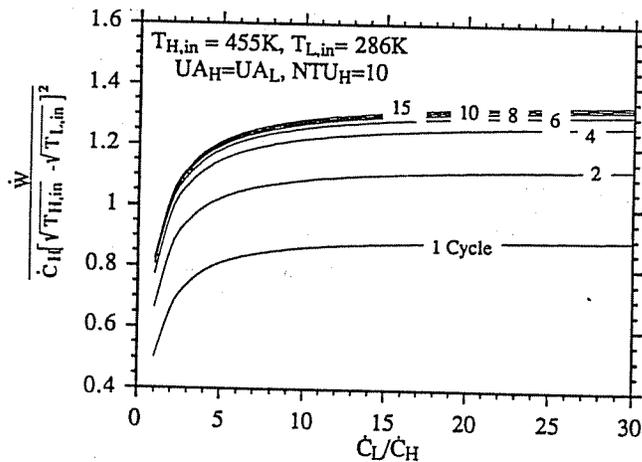


Fig. 13 Variation of maximum power with heat capacitance rate ratio

$$T_{L,in,i-1} = T_{L,in,i} + \epsilon_L(T_{H,i} - T_{L,in,i}) \quad (56)$$

$T_{H,in,i}$ and $T_{L,in,i}$ are the source and sink inlet temperatures for a Carnot cycle in the sequence where $T_{H,in,1}$ and $T_{L,in,N}$ are specified inlet source and sink temperatures. As the cycles in the sequence extract and reject heat, the source temperature decreases and sink temperature increases in the flow direction. $\epsilon_{H,i}$ and $\epsilon_{L,i}$ are the effectivenesses of the hot-side and cold-side heat exchangers of each cycle, which are assumed to be equal in this analysis.

An analytical solution is not apparent for this optimization problem. However, the optimum heat power cycle with finite thermal capacitance rate heat source and heat sink can be determined numerically. Consider a hot fluid stream entering the boiler at 455 K, and a cold fluid stream entering the condenser heat exchanger at 286 K.

The required number of Carnot cycles in sequence that will sufficiently identify the cycle is considered for the case where $NTU_H=10$ and $C_L/C_H=10$. Figure 12 shows the efficiency at maximum power as the number of Carnot cycles in sequence increases from 1 to 15. The efficiency at maximum power is approximately independent of the number of Carnot cycles in the sequence. The efficiency of the optimum heat power cycle is almost the same as the maximum power efficiency of a single Carnot cycle operating between the same external streams (the 1 percent difference noted in this case can be due to the numerical techniques and roundoff error). However, the maximum power increases significantly as the number of Carnot cycles in sequence increases from 1 to 15. The difference between the maximum power obtained from 10 cycles in sequence and 15 cycles in sequence is very small (less than 1 percent),

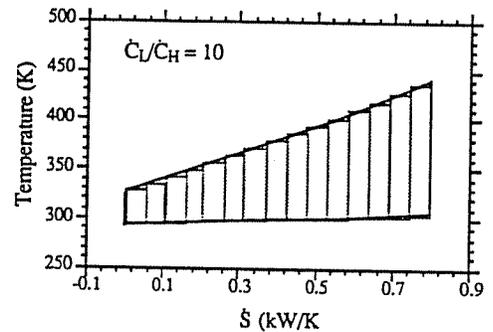
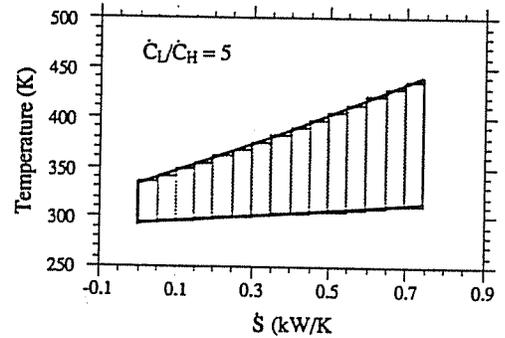
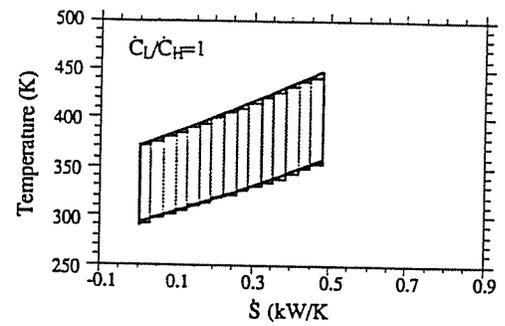


Fig. 14 Variation of the maximum power cycle shape with heat capacitance rate ratio for $T_{L,in}=286$ K, $T_{H,in}=455$, and $UA_H=UA_L$ ($NTU_H=10$)

which indicates that the 15 cycles approximate the optimum heat power cycle.

The effect of thermal capacitance rates ratio (C_L/C_H) on the maximum power for different number of Carnot cycles in sequence is shown in Fig. 13. As the thermal capacitance rate ratio increases, the power output increases rapidly at first and then levels off for thermal capacitance rates ratios greater than 5 approaching an asymptotic limit. However, there is no significant variation in the power output as C_L/C_H is increased above 10.

Figure 14 shows the shape of the optimum heat power cycles in a $T-S$ diagram. The slopes of the heating and cooling processes, as well as the cycle temperatures, vary with the thermal-capitance rate ratio. Similar graphs can be constructed to show the effects of heat exchanger conductances on the optimum power cycle shape. The shape of the optimum heat power cycles varies from rectangular for infinitely small heat exchangers to triangular for infinitely large heat exchangers.

Conclusions

The optimum power produced by the closed Brayton, Otto, and other familiar cycles can be greater than the optimum power produced by the Carnot cycle for the same working conditions and heat exchanger conductances. The heat power cycle that produces the maximum power for specified boundary conditions provides a useful tool for studying power cycles and forms the basis for making design improvements. The shape of the optimum heat power cycle varies with the external

boundary conditions such as the type of heat source and heat sink, and heat exchanger conductance.

If the heat source and sink have infinite heat capacitance rates (i.e., isothermal), the heat transfer processes for the optimum cycle are isothermal, i.e., the Carnot cycle is the optimum cycle. However, if the heat source and the heat sink have finite heat capacitance rates, the heat transfer processes for the optimum cycle are not isothermal, but rather occur over a range of temperatures. This variable temperature at both the cooling and heating processes can be achieved either by varying the pressure during the phase change of a pure fluid or by keeping the pressure constant during the phase change of a nonazeotropic binary mixture. The thermodynamic advantages of the cycles proposed by Kalina (e.g., Kalina, 1983), which uses ammonia-water as working fluid may be explained in this manner.

For both finite and infinite capacitance rates, the efficiency at maximum power of internally reversible Carnot and Brayton cycles is equal to $1 - \sqrt{T_{L,in}/T_{H,in}}$. Moreover, the efficiency of the optimum heat power cycles is found to be well approximated by the same form. Because this efficiency depends only on the inlet temperatures of the heat reservoirs, it can be obtained easily and used as a design tool. In the case of internal irreversibility, the efficiency at maximum power is no longer given by this simple form. Rather it is given by a more complicated form, which depends on how the internal irreversibility is modeled. In this study, a simple model relating the entropy change during heat rejection and heat addition is used. The result is a modification to the efficiency at maximum power, which is found to be equal to or well approximated by $\eta = 1 - \sqrt{T_{L,in}/\phi T_{H,in}}$, where ϕ is a factor relating the entropy changes during heat rejection and heat addition.

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